Reading Questions 22

page 132: Definition 6.6

page 133: Definition 6.8

$(x+y)^n$

$\binom{n}{k} = \frac{n!}{k!(n-k)!}$

1. A number $\binom{3}{2}$ is a binomial coefficient. T
2. A number $\binom{3}{2}$ is $\frac{\cancel{3}!}{\cancel{3}!(3-2)!}$. F
3. Write the element of $\binom{[3]}{2}$? $\{\{1,2\},\{1,3\},\{2,3\}\}$

Section 6.1 The Fundamental Counting Principle (Part 1)

FCP

P 1. Let G be a group such that $x, y \in G$. Let H be a subgroup of G. Show that $|xH| = |yH|$.

P 2. Does the proof of Lagrange's theorem work for right cosets?

$g \cdot Hx = Hgx$ works for right cosets.

Thm: (FCP)

Let the group G act on the set Ω such that $\alpha \in \Omega$.

Then $|O_\Omega(\alpha)| = |G : \mathrm{Stab}_G(\alpha)|$

pf: We know $\mathrm{Stab}_G(\alpha) \leq G$

Let X be the distinct left cosets of $\mathrm{Stab}_G(\alpha)$.

Define $\lambda : X \to O_\Omega(\alpha)$ such that

$\lambda(x\,\mathrm{Stab}_G(\alpha)) = x \cdot \alpha$ where $x \in G$.

WTS λ is a 1-1 correspondence.

Suppose $\lambda(x\,\mathrm{Stab}_G(\alpha)) = \lambda(y\,\mathrm{Stab}_G(\alpha))$

$$\Rightarrow \quad x \cdot \alpha = y \cdot \alpha$$

$$\Rightarrow \quad x^{-1} \cdot x \cdot \alpha = x^{-1} \cdot y \cdot \alpha$$

$$\Rightarrow \quad x^{-1}x \cdot \alpha = x^{-1}y \cdot \alpha$$

$$\Rightarrow \quad e \cdot \alpha = x^{-1}y \cdot \alpha$$

$$\Rightarrow \quad \alpha = x^{-1}y \cdot \alpha$$

$$\Rightarrow \quad x^{-1}y \in \operatorname{stab}_G(\alpha)$$

$$\Rightarrow \quad x^{-1}y = h \quad \text{where} \quad h \in \operatorname{stab}_G(\alpha)$$

$$\Rightarrow \quad y = xh$$

$$\Rightarrow \quad y \in x \operatorname{stab}_G(\alpha)$$

$$\Rightarrow \quad y \operatorname{stab}_G(\alpha) = x \operatorname{stab}_G(\alpha)$$

This shows that λ is 1-1.

Suppose $\beta \in O_\Omega(\alpha)$. Then $\exists\, x \in G$ s.t.

$x \cdot \alpha = \beta$. So $\lambda(x \operatorname{stab}_G(\alpha)) = x \cdot \alpha = \beta$

This show λ is onto.

$\therefore$ λ is a bijection and $|X| = |O_G(\alpha)|$ and

$$|O_\Omega(\alpha)| = |G : \operatorname{stab}_G(\alpha)|$$

<u>cor:</u> (Lagrange's thm)

Let G be a finite group s.t. $H \leq G$. Then

$$|G| = |H| \cdot |G : H|.$$

pf: Let Ω be the set of distinct left cosets of H in G. Then G acts on Ω by $g \cdot xh = gxH$ where $g, x \in G$.

Let $yH, xH \in \Omega$ where $yH \neq xH$.

WTS. $\exists_{g \in G}$ s.t. $gyH = xH$. Let $g = xy^{-1}$. Then $xy^{-1} \cdot yH = xy^{-1}yH = xH$. This shows $\forall_{\alpha \in \Omega}$

$$O_\Omega(\alpha) = \Omega \left(\text{in other word } O_\Omega(xH) = \Omega \right).$$

Thus $|O_\Omega(H)| = |\Omega|$

Moreover

$$|O_G(H)| = |G:H| \qquad (1)$$

Now

$$\begin{aligned} stab_G(H) &= \{ g \in G : g \cdot H = H \} \\ &= \{ g \in G : gH = H \} \\ &\Rightarrow g \in H \\ &= H. \end{aligned}$$

Thus $|stab_G(H)| = |H|$. $\qquad (2)$

$\therefore$ FCP $\quad |O_G(H)| = |G : stab_G(H)|$

finite $\quad |G| = |O_G(H)| \cdot |stab_G(H)|$

$$= |G:H| \cdot |H| \qquad \text{by (1) \& (2)}.$$

lem: Let n and k be positive integers where $n \geq k$. Then $\binom{n}{k} = \dfrac{n!}{k!\,(n-k)!}$.

pf: Let $G = S_n$ and $\Omega = \binom{[n]}{k}$. want to find $|\Omega|$!

we know

Define the group action $|\mathcal{O}_\Omega(\alpha)| = |G : \mathrm{Stab}_G(\alpha)|$

$$\sigma \cdot \{x_1, \dots, x_k\} = \{\sigma(x_1), \sigma(x_2), \dots, \sigma(x_k)\} \in \Omega$$

where $\sigma \in G = S_n$ $x_i \in [n]$

Let $\{x_1, \dots, x_k\}, \{y_1, \dots, y_k\} \in \Omega$.

$$\underbrace{(x_1 y_1)(x_2 y_2)\cdots(x_k y_k)}_{\sigma} \cdot \{x_1, \dots, x_k\} =$$

$$= \{\sigma(x_1), \dots, \sigma(x_k)\}$$

$$= \{y_1, \dots, y_k\}$$

$\sigma(x_i) = y_i$

This shows $\mathcal{O}_\Omega(\{x_1, \dots, x_k\}) = \Omega$

$$\Rightarrow |\mathcal{O}_\Omega(\{x_1, \dots, x_k\})| = |\Omega| = \binom{n}{k}$$

By FCP $\binom{n}{k} = |G : \mathrm{stab}_G(\{x_1, \dots, x_k\})|$

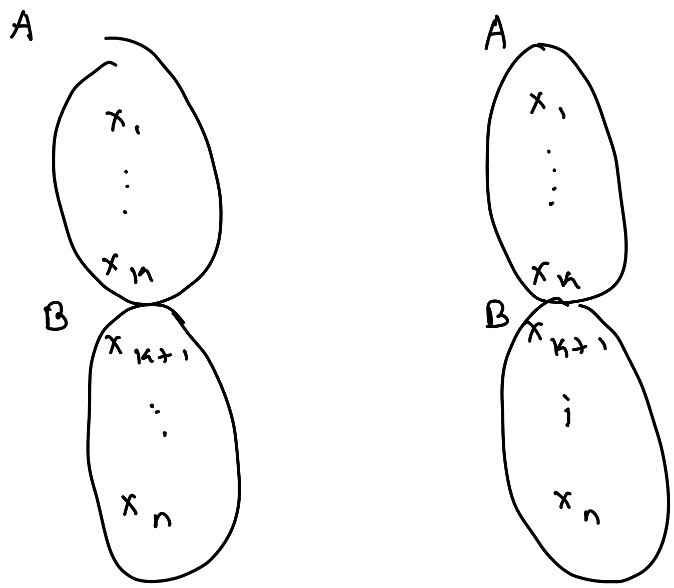
# of distinct left cosets of $\mathrm{stab}_G(\{x_1, \dots, x_n\})$

for the finite

$$n! = |S_n| = |G| = |O_\Omega\{x_1, \ldots, x_k\}| \cdot |\mathrm{Stab}_G(\{x_1, \ldots, x_k\})|$$

$$= \binom{n}{k} |\mathrm{Stab}(\{x_1, \ldots, x_k\})|$$

$$\Rightarrow \quad \binom{n}{k} = \frac{n!}{|\mathrm{Stab}_G(\{x_1, \ldots, x_k\})|}$$



there are $k!$ bijections from A to A

and

there are $(n-k)!$ bijections from B to B

$\sigma \in \mathrm{Stab}_G(\{x_1, \ldots, x_k\})$ iff σ maps A to A and maps B to B. Therefore $|\mathrm{Stab}_G(\{x_1, x_2, \ldots, x_k\})|$

$= k!\,(n-k)!$

$$\therefore \quad \binom{n}{k} = \frac{n!}{k!\,(n-k)!}$$