Reading Questions 21

page 115: Theorem 5.13 and its proof

1. Let G be a group such that H is a subgroup of G. Then $|G|$ divides $|H|$. F

2. Let G be a group such that H is a subgroup of G. What is $|G:H|$?

# of right cosets ↑ index

Section 5.2 Lagrange Theorem (Part 1)

Lagrange Results

P 1. Let $G = S_5$ and $H =< (12) >$. What is $|G:H|$?

P 2. Let G be a group such that $|G| = n$. Prove $a^n = e$. $a \in G$

P 3. Let G be a finite group such that $H \leq K \leq G$. Prove $|G:K| \cdot |K:H| = |G:H|$.

5.2

lem Let G be a group such that $H \leq G$ and $g \in G$. Then $|H| = |Hg|$.

pf: Define $\sigma : H \to Hg$ where $\sigma(h) = hg$.

Then σ is clearly well defined.

Suppose $\sigma(h_1) = \sigma(h_2)$. Then $h_1 g = h_2 g \Rightarrow h_1 = h_2$.

This shows that σ is 1-1.

Let $y \in Hg$. Then by def of Hg $\exists \hat{h} \in H$ s.t $\hat{h} g = y$.

Also $\sigma(\hat{h}) \overset{\text{def of } \sigma}{=} \hat{h} g = y$. This shows that σ is onto.

$\therefore$ σ is a 1-1 correspondence and $|H| = |Hg|$.

cor: $|Hg_1| = |Hg_2|$

pf: $|Hg_1| = |H|$ & $|Hg_2| = |H| \Rightarrow |Hg_1| = |Hg_2|$

Thm: (Lagrange's theorem) Let G be a finite group such that $H \leq G$.

Then $|G:H| = |G|/|H|$ or $|G| = |H| \cdot |G:H|$.

pf: We know G/H partitions G. Hence

$$|G| = |Hg_1| + |Hg_2| + \cdots + |Hg_{|G:H|}|$$ Hg_i are the distinct right cosets

$$= \underbrace{|H| + |H| + \cdots + |H|}_{|G:H|}$$ by previous lem

$$= |G:H|\,|H|$$

cor: Let G be a finite group such that $a \in G$.

Then $o(a)$ divides $|G|$.

pf: $\langle a \rangle \leq G$ so by previous thm $\underbrace{|\langle a \rangle|}_{= o(a)} \,\big|\, |G|$.

cor: Let G be a group such that $|G|$ is prime.

Then $G \cong \mathbb{Z}_p$.

pf: WTS G is cyclic. Let $e \neq x \in G$.

Consider $\langle x \rangle$. By lagrange's thm

$|G|/|\langle x\rangle| = |G:\langle x\rangle| \in \mathbb{Z}$. Since $|G|$ is prime

$|G|/|\langle x\rangle|$ is an integer $|\langle x\rangle| = p, 1$.

$|\langle x\rangle|$ is not 1 since $x \neq e$, $\therefore$ $|\langle x\rangle| = p$.

and $G = \langle x\rangle$ ($\langle x\rangle \leq G$ and $|\langle x\rangle| = |G|$)

<u>Thm:</u> Let G be a finite group such that $H, K \leq G$,

Then $|G:K| = |H : H \cap K|$ iff <u>$G = HK$</u>.