Reading Questions 20

page 110: Definition 5.1

1. Let G be a group such that $x \in G$ and $H \subseteq G$. If Hx is a right cosets then H must be a subgroup of G. T
2. Let G be a group such that $x \in G$ and $H \leq G$. Then Hx is a subgroup of G. F
3. Let $G = \mathbb{Z}_5$ and $H =< 2 >$ and $x = 3$. List the elements of Hx. $= \{0, 2, 4, 1, 3\} \cdot 3 = \{0, 1, 2, 3, 4\}$

$H3 =$

Section 5.1 Translation Action and Cosets (Part 1)

$|Hx| = |H|$

Cosets

P 1. Let $G = S_4$ and $H =< (123) >$. List the right cosets of H in G.

P 2. Let G be a group such that $H \leq G$ and $x, y \in G$. Prove $Hx = Hy$ if and only if $yx^{-1} \in H$.

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P 3. Let $G = S_4$ and $H =< (123) >$. What is $|H : G|$?

P 4. Let $G = D_8$ and $H =< R_{90} >$. List the left cosets of H in G.

5.1

Def: $Hx = \{ hx : h \in H \}$ - Right coset of H in G where $H \leq G$.

Ex: Let $G = D_8$ and $H = \langle R_{180} \rangle$. Then

$H = \{R_0, R_{180}\}$ $\quad HR_{90} = \{R_0 \cdot R_{90}, R_{180} \cdot R_{90}\}$

$= \{R_{90}, R_{270}\}$

Note: $R_0 \notin HR_{90} \Rightarrow Hx$ is not a subgroup of G.

Cor: The distinct right cosets partition the group.

pf: The right cosets are orbits.

$H \leq G$ $\quad$ $H \overset{\text{action}}{\curvearrowright} G$

Ex: Let $G = S_3$ and $H = \underset{\{(1),(12)\}}{\langle(12)\rangle}$. Then

$H(1) = H = H(12) = \{(1), (12)\} = O_H((12))$ $\quad$ H acts on G

$= \{x \in G : h \cdot (12) = x\}$ $\quad$ $h \cdot g = hg$

$x \sim (12)$

$H(13) = \{(1)(13), (12)(13)\} = \{(13), (132)\}$

$= H(132) = \{(1)(132), (12)(132)\}$

$= O_H(13) = O_H(132)$

$H(23) = \{(1)(23), (12)(23)\} = \{(23), (123)\} = H(123)$

$= O_H(12)$

$= O_H(123)$

$|Hx| = |Hy|$

lem: Let G be a group such that $H \leq G$ and $x, y \in G$.

(1) $Hx = Hy \iff y \in Hx$

pf: $(\Rightarrow)$ Suppose $Hx = Hy$. Since $H \leq G$ $e \in H$. Hence

$y \in Hy$ since $e \cdot y \in Hy \Rightarrow y \in Hy$. $\therefore$ $y \in Hx$ since $Hx = Hy$.

($\Leftarrow$) Let $y \in Hx$. Let $z \in Hx$. Then $\exists\, h_1, h_2, \in Hx$ s.t. $z = h_1 x$ and $y = h_2 x$. Hence $x = h_2^{-1} y$

$\Rightarrow z = h_1 \underbrace{h_2^{-1} y}_{x}$ and $h_1 h_2^{-1} \in H \Rightarrow z \in Hy \quad \therefore \; Hx \subseteq Hy$

Let $z \in Hy$. Then $\exists\, h_1, h_2 \in H$ s.t. $z = h_1 y$ and $y = h_2 x$. Hence $z = h_1 h_2 x \Rightarrow z \in Hx$.

$\therefore \; Hx = Hy$

cor: $Hg = H \iff g \in H$

Def: Let G be a group such that $H \leq G$. The set of all distinct right cosets is G/H. The number of distinct right cosets is $|G/H|$ or $|G:H|$ is the index of H in G.

Ex: From the previous example $|S_3 : \langle 12 \rangle| = 3$

lem: Let G be a group such that $H \leq G$. Then

(1) $|G : \{e\}| = |G|$

(2) $|G : G| = 1$

(3) $|G : H| := \underbrace{\#\text{ of left cosets of } H}_{xH}$

<u>lem 5.9</u> $\forall g \in G \quad H \leq G$

$$|Hg| = |H|$$