Reading Questions 18

page 96: Definition 4.24,4.26,4.27

1. A relation of a set is a set. T
2. An equivalence relation of a set is a function. F
3. Consider the relation $\{\{1,2\},\{2,1\},\{1,3\}\}$ on the set $\{1,2,3\}$. What element is missing from the relation that would make the relation symmetric?

$\{3,1\}$

Section 4.4 Orbits (Part 1)

Equivalence Relations

P 1. Prove or disprove: Let $a \sim b$ if $a, b \in \mathbb{Z}$ and $a \leq b$. Then $\sim$ is an equivalence relation on $\mathbb{Z}$.

P 2. Let $a \sim b$ if $a, b \in \mathbb{Z}$ and $a \leq b$. Find cl(2).

Orbits

P 3. Let $G = S_7$. Let $H =< (23), (132) >$ act on $\Omega = [7]$ where $h \cdot a = h(a)$ for $h \in H$ and $a \in \Omega$. What are the orbits of Ω?

P 4. Let $\mathrm{GL}(n, \mathbb{R})$ act on $\mathrm{M}_{n\times n}(\mathbb{R})$ where $P \cdot A = PAP^{-1}$ for $P \in \mathrm{GL}(n, \mathbb{R})$ and $A \in \mathrm{M}_{n\times n}(\mathbb{R})$. What are the orbits of A?

P 5. Let $\mathrm{GL}(n, \mathbb{R})$ act on $\mathrm{M}_{n\times n}(\mathbb{R})$ where $P \cdot A = PA$ for $P \in \mathrm{GL}(n, \mathbb{R})$ and $A \in \mathrm{M}_{n\times n}(\mathbb{R})$. What are the orbits of A?

<u>Def:</u> An equivalence relation on a non-empty set X is a relation R that has the following property.

(1) $\forall a \in X \quad (a,a) \in R$

(2) $\forall a,b \in X \quad (a,b) \in R \quad (b,a) \in R$

(3) $\forall a,b,c \in X \quad (a,b), (b,c) \in R \quad (a,c) \in R$

<u>Ex:</u> $\leq$ consider $\mathbb{Z}$ and $\mathbb{Z} \times \mathbb{Z}$

$(1,2) \in R_{\leq}$ since $1 \leq 2$ $\qquad a \leq a$

$(2,1) \notin R \qquad 2 \nleq 1 \qquad \leq$ is not an eq. relation.

Ex: Let X be the set of points in a plane. For $x, y \in X$ $\quad x \sim y$ if x and y are the same distance from the origin. Then $\sim$ is an equivalence relation.

$$X = \mathbb{R}^2 = \{(x_1, x_2) : x_1, x_2 \in \mathbb{R}\}$$

$$\sim := R_K = \{((x_1, x_2), (x_3, x_4)) : \sqrt{x_1^2 + x_2^2} = \sqrt{x_3^2 + x_4^2} = K\}$$

$$\subseteq X \times X = \mathbb{R}^2 \times \mathbb{R}^2$$

(reflexive) if $(x_1, x_2) \in R_K$ then $\sqrt{x_1^2 + x_2^2} = \sqrt{x_1^2 + x_2^2}$

$$(x_1, x_2) \sim (x_1, x_2)$$

(symmetric) $(x_1, x_2) \sim (x_2, x_1)$ since $\sqrt{x_1^2 + x_2^2} = \sqrt{x_2^2 + x_1^2}$

(transitive) suppose $(x_1, x_2) \sim (y_1, y_2)$ & $(y_1, y_2) \sim (z_1, z_2)$

then $\sqrt{x_1^2 + x_2^2} = \sqrt{y_1^2 + y_2^2}$ & $\sqrt{y_1^2 + y_2^2} = \sqrt{z_1^2 + z_2^2}$

$$\Rightarrow \sqrt{x_1^2 + x_2^2} = \sqrt{z_1^2 + z_2^2}$$

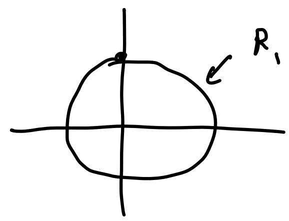
$$\Rightarrow (x_1, x_2) \sim (z_1, z_2)$$

Def: Let R be a relation on X. Let $a \in X$. Then

$cl(a) = \{x \in X : (a, x) \in R\}$: class of a.

If R is an eq. relation then $cl(a)$ is the equivalence class of a.

Ex: From the previous example $cl((0,1)) = R_1$



lem: Let $\sim$ be an eq. relation on X. Then the eq. classes partition X.

pf: see board

lem: Let G act on Ω. Let $\alpha, \beta \in \Omega$. Define

$\alpha \sim \beta$ if $\exists g \in G$ such that $g \cdot \alpha = \beta$.

Then $\sim$ is an equivalence relation on Ω.

pf:

(reflexive) $e_G \cdot \alpha = \alpha \implies \alpha \sim \alpha$

(symmetric) suppose $\alpha \sim \beta$

$\exists g \in G$ s.t $g \cdot \alpha = \beta$

$$g^{-1} \cdot g \cdot \alpha = g^{-1} \cdot \beta$$

$$\implies g^{-1} g \cdot \alpha = g^{-1} \cdot \beta$$

$$\implies e_G \cdot \alpha = g^{-1} \cdot \beta$$

$$\implies \alpha = g^{-1} \cdot \beta$$

(transitive) Suppose $\alpha \sim \beta$ & $\beta \sim \tau$.

$\exists g, h \in G$ s.t $g \cdot \alpha = \beta$ & $h \cdot \beta = \tau$

$$hg \cdot \alpha = h \cdot g \cdot \alpha = h \cdot \beta = \tau$$

Def: Let G act on Ω. The orbit of α is

$$\begin{aligned} O_\Omega(\alpha) = G\alpha &= \{ \beta \in \Omega : \exists g \in G \text{ with } g \cdot \alpha = \beta \} \\ &= \{ \beta \in \Omega : \alpha \sim \beta \} \\ &= cl(\alpha) \end{aligned}$$

Thm: Let G act on Ω. Then the orbits partition Ω.

Ex: Let S_8 act on $\overset{\Omega}{\overset{\shortparallel}{[8]}}$ where $g \cdot \alpha = g(\alpha)$.

consider

$$\langle (56), (123) \rangle = H = \{ e, (123), (56), (123)(56), (132), (132)(56) \}$$

Then

$$\begin{aligned} \text{stab}_H(1) &= \{e, (56)\} \\ &= \text{stab}_H(2) \\ &= \text{stab}_H(3) \end{aligned}$$

$$\text{stab}_H(4) = H = \text{stab}_H(7) = \text{stab}_H(8)$$

$$\begin{aligned} \text{stab}_H(5) &= \{e, (123), (132)\} \\ &= \text{stab}_H(6) \end{aligned}$$

and

$$O_\Omega(1) = \{1, 2, 3\} = O_\Omega(2) = O_\Omega(3) \qquad O_\Omega(5) = \{5, 6\} = O_\Omega(6)$$

$$O_\Omega(4) = \{4\}, \quad O_\Omega(7) = \{7\}, \quad O_\Omega(8) = \{8\}$$

conjeture: $|H| = |\mathrm{stab}_H(\alpha)| \cdot |O_\Omega(\alpha)|$