Reading Questions 15

page 86: Definition 4.1

1. A group action is a set Ω. F
2. A group action is a homomorphism σ. T
3. Suppose G acts on the set Ω. Let a be the identity element in G and $\omega \in \Omega$. What is $a \cdot \omega$? $a \cdot \omega = \omega$

Section 4.1 Group Actions (Part 1)

Definition and Examples

P 1. Let $\Omega = \{\{1,2\},\{1,3\},\{1,4\},\{2,3\},\{2,4\},\{3,4\}\}$. Let $\sigma = (123)$ $\in S_4$. Then S_4 acts on Ω where $\sigma \cdot \{a,b\} = \{\sigma(a),\sigma(b)\}$. Compute $\sigma \cdot \{1,4\}$ and $\sigma \cdot \{2,3\}$.

$\{\sigma(1), \sigma(4)\} = \{2, 4\}$

P 2. Find a subgroup of S_4 which is isomorphic to Z_4. Hint Z_4 acts on $\{0,1,2,3\}$ where $g \cdot a = g + a \bmod 4$.

$\langle (1234) \rangle$ $\quad H = \{\sigma_0, \sigma_1, \sigma_2, \sigma_4\}$

$0 \cdot 0 = 0$
$0 \cdot 1 = 1$ $\quad (0)(1)(2)(3)$
$0 \cdot 2 = 2$ $\quad (1)(2)(3)(4)$
$0 \cdot 3 = 3$ $\quad \sigma_0 = (1)$

P 3. Find a subgroup of S_4 which is isomorphic to D_6. Hint D_6 acts on $\{1,2,3\}$.

P 4. Let $G = GL(n,\mathbb{R})$ and let Ω be the set of all real $n \times n$ matrices. Let $A \in G$ and $B \in \Omega$. Define $A \cdot B = BAB^{-1}$. Show that G acts on Ω.

P 5. Let G be a group such that $H \leq G$. Prove or disprove: H acts on G where $h \cdot g = gh^{-1}$.

$1 \cdot 0 = 1+0 = 1$
$1 \cdot 1 = 1+1 = 2$
$1 \cdot 2 = 1+2 = 3$
$1 \cdot 3 = 1+3 = 0$

4.1

$0 \to 1$
$1 \to 2$
$2 \to 3$
$3 \to 4$

$\sigma_1 = (0123)$
(1234)

Def: Let G be a group and let Ω be a set.

Then G acts on Ω if $\forall g \in G$ and $\forall \alpha \in \Omega$

then $g \cdot \alpha = g(\alpha) \in \Omega$ such that

(1) $\forall \alpha \in \Omega$ $\quad e \cdot \alpha = \alpha$

(2) $\forall g, h \in G$ $\quad g \cdot (h \cdot \alpha) = gh \cdot \alpha$

$\lambda : \Omega \to \Omega$

Ex: Let A be a set. Suppose we can construct the group G.

$$A \to G \implies \forall g \in G \quad g: A \to A$$

Defined $h: \Omega \to A$. Then G acts on Ω

$\uparrow$
bijection

Ex: S_n acts on $[n]$. Let $\sigma \in S_n$ and $i \in [n]$.
$ \| \; \Omega$

Define $\sigma \cdot i = \sigma(i)$.

(1) $\sigma \cdot i \in \Omega$ cleary, well $\sigma : [n] \to [n]$

(2) $e \cdot i = e(i) = i$ ✓

(3) let $g, h \in S_n$. Then $g \cdot h \cdot i = g \cdot h(i)$
$= g(h(i))$

$$g(h(i)) = gh(i)$$

$$\therefore \quad g \cdot h \cdot i = gh(i) = gh \cdot i$$

Hence $\sigma \cdot i$ is a group action lem: Let G act on Ω such that $\forall g \in G$ and $\forall a \in \Omega$

$g \cdot a = \sigma_g(a)$ where $\sigma_g : \Omega \to \Omega$

Then σ_g is a bijection.

pf: First $\forall b \in \Omega$, $\sigma_g(b) = g \cdot b \in \Omega$ as G acts on Ω.

This shows σ_g is a map.

Now suppose $\sigma_g(x_1) = \sigma_g(x_2)$. Then

$$g \cdot x_1 = g \cdot x_2 \Rightarrow g^{-1} \cdot g \cdot x_1 = g^{-1} \cdot g \cdot x_2$$

$$\Rightarrow g^{-1}g \cdot x_1 = g^{-1}g \cdot x_2 \quad G \text{ acts on } \Omega$$

$$\Rightarrow e \cdot x_1 = e \cdot x_2 \quad G \text{ is a group}$$

$$\Rightarrow x_1 = x_2 \quad G \text{ acts on } \Omega$$

Hence σ_g is 1-1.

Let $c \in \Omega$. Then $e \cdot c = c \Rightarrow gg^{-1} \cdot c = c$ G is a group

$\Rightarrow g \cdot g^{-1} \cdot c = c$ G acts on Ω

Let $g^{-1} \cdot c = b \in \Omega$

$\Rightarrow g \cdot b = c$.

$\Rightarrow \sigma_g(b) = c$

$\therefore$ σ_g is onto and a bijection.

Ex: D_8 acts on $[4]$.

$e \to (1)$

$a \to (1\,2\,3\,4)$

$a^2 \to (1\,3)(2\,4)$

$a^3 \to (1\,4\,3\,2)$

$b \to (1\,2)(3\,4)$

$ab \to (1\,3)$

$a^2 b \to (1\,4)(2\,3)$

$a^3 b \to (2\,4)$

$\therefore \; \{ (1), (1234), (13)(24), (1432), (12)(34), (13), (14)(23), (24) \}$

$\leq S_4$.

Ex: $GL(2, \mathbb{R})$ acts on $\mathbb{R}^2$ where $\forall A \in GL(2,\mathbb{R})$

and $\forall \vec{x} \in \mathbb{R}^2$ $\quad A \cdot \vec{x} = A\vec{x} \in \mathbb{R}^2$

(1) $\underset{\begin{bmatrix}1&0\\0&1\end{bmatrix}}{I} \cdot \vec{x} = I\vec{x} = \vec{x}$ ✓

(2) $A \cdot B \cdot \vec{x} = A \cdot B\vec{x} = \underbrace{AB}\vec{x}$

$= AB \cdot \vec{x}$