Reading Questions 14

page 78: Definition 3.5

1. Let $\sigma \in S_n$. Then σ is an even permutation if $o(\sigma)$ is even. T
2. The cycles $(12) \in S_4$ is a odd permutation. T
3. Is the permutation $(12)(24)(12) \in S_4$ even or odd? odd

Section 3.2 Alternating Groups (Part 1)

Properties

P 1. Write $(123)(2345)(321)$ as a product of 2−cycles.

P 2. Let $\sigma, \tau \in S_n$. Prove or disprove. If σ and τ are both odd then $\sigma\tau$ is even.

P 3. List the elements of A_4.

P 4. Determine if $(123)(4567) \in A_8$.

P 5. Prove that A_n can be generated by all 3−cycles in S_n.

Recall

$(2345) \in S_7$ can be written as a product of 2-cycles.

$(2345) = (25)(24)(23)$ method 1

$= (23)(34)(45)$ method 2

(2345) - is an odd cycle

lem: Let $\sigma, \tau \in S_n$. If σ and τ are both even or odd then $\sigma\tau$ is even.

pf: (even) $\exists k, r \in \mathbb{Z}$ $\quad i, j$ are 2-cycle

$\sigma = i_1 i_2 \cdots i_{2k} \qquad \tau = j_1 \cdots j_{2r}$

$\sigma\tau = i_1 i_2 \cdots i_{2k} j_1 \cdots j_{2r}$ which has $\overbrace{\text{an even \# of}}^{2k+2r = 2(k+r)}$ 2-cycle

Def: Let n be a positive integer and greater than 1 and A_n be the set of all even permutation of S_n.

Ex: $A_3 = \{ \underset{e}{\underset{\shortparallel}{(12)(12)}} , (12)(23) \}$

Thm: $A_n \leq S_n \quad n > 1$

pf: $A_n \subseteq S_n$ clearly!

1) $(12)(12) = e$ Hence $e \in A_n$

2) associativity follows from S_n

3) $\sigma, \tau \in A_n$ then σ, τ are even. By the previous $\sigma\tau$ is even. Hence A_n is closed.

4) $\sigma = (i_1 i_2)(i_1 i_3) \cdots (i_1 i_m)$ $\quad m$ - is even

then $\sigma^{-1} = (i_1 i_m)(i_1 i_{m-1}) \cdots (i_1 i_3)(i_1 i_2)$

since $\underline{\sigma\sigma^{-1}} = \underline{\sigma^{-1}\sigma} = e$

$\therefore$ A_n is a group and $A_n \leq S_n$

Thm: No permutation is both even and odd

cor: Let n be an integer greater than 1. Then

$$|A_n| = \frac{1}{2}|S_n| = \frac{n!}{2}.$$