Reading Questions 13

page 75: Definition 3.1

1. A cycle in S_n of length m is a $m-$ cycle. T
2. The cycles (123) and (231) are disjoint cycles. F
3. What is the length of (123)(24). (124) -> 3

Section 3.1 Permutations, Cycles, and Transpositions (Part 1)

Cycles and Transpositions

P 1. List all 3 cycles in S_4.

P 2. Give an example of 2 disjoint transpositions in S_6.

P 3. If σ is a transposition, what is σ^{-1}?

P 4. Prove the following statement. Let n be a positive integer. If σ and τ are disjoint cycles in S_n then $\sigma\tau = \tau\sigma$.

Properties

P 5. Write (123)(24)(321) as a product of disjoint cycles.

P 6. Write (1234)(231) as a product of transpositions.

P 7. A simple transposition is a transposition of the form $(i\, i+1)$. Write (1245) as a product of simple transpositions.

P 8. What is the order of (123)(25)(46) in S_7?

3.1

Recall $[n] := \{1, 2, \dots, n\}$

$S_n = \text{Perm}([n]) =$ all bijections from $[n]$ to $[n]$

Def: Let $\sigma \in S_n$ such that $\sigma = (i_1, i_2, \dots, i_m)$ where $i_1, i_2, \dots, i_m$ are distinct. Then σ is an m-cycle or a cycle of length m.

Ex: Let $\sigma = (1\,2\,3) \in S_5$. Then σ is a 3-cycle.

Def: The cycles $(i_1, \dots, i_m)$ and $(j_1, \dots, j_\ell)$ are disjoint if $\{i_1, \dots, i_m\} \cup \{j_1, \dots, j_\ell\} = \{\}$.

Ex: The cycles $(1\,2\,3)$ and $(2\,4\,5)$ are not disjoint while $(1\,2\,3)$ and $(4\,5)$ are disjoint cycles.

Def: A 2-cycle is called a transposition.

lem: Let n be a positive integer.

(1) Every element in S_n can be written as a product of disjoint cycles.

pf: obvious since $\forall \sigma \in S_n$ σ is a bijection.

(2) Every element in S_n can be written as a product of transpositions.

pf: $\sigma = (i_1, i_2, \dots, i_m)$ $\quad i_j \to i_{j+1}$ $j < m$ $\quad i_m \to i_1$

Then $\beta = (i_1 i_m)(i_1 i_{m-1}) \cdots (i_1 i_2)$

$\beta(i_1) = i_2 \quad \beta(i_2) = i_3 \quad \beta(i_j) = i_{j+1} \quad j < m$

$\beta(i_m) = i_1$

Ex: $(1\,2\,3)(3\,4\,5) = (2\,3\,4\,5) = (2\,5)(2\,4)(2\,3)$

$$\parallel$$

$(13)(12)(35)(34)$

Thm: The order of a permutation σ is lcm of the length of the cycle in a cycle decomposition of σ.

Ex: what is the order of $\sigma = (123)(25) = (1253)$? Hence $o(\sigma) = 4$ by the previous thm.

Ex: What is the order of $\sigma = (123)(45)$? σ is a cycle decomposition so $o(\sigma) = \text{lcm}(3,2) = 6$

$(1253)^4 = e$

$(1253)^2 = (15)(23)$

$(1253)^3 = (1352)$

$1 < i <$ length of σ
$\Rightarrow \sigma(i) \neq i$

Hence if σ is a cycle then $o(\sigma) =$ length of σ

if σ, τ are disjoint then $\sigma\tau = \tau\sigma$

$((123)(45))^6 = e$

$$\begin{aligned}((123)(45))^2 &= (123)(45)(123)(45) \\ &= (123)(123)(45)(45) \\ &= (123)^2(45)^2 \\ &= (123)^2\end{aligned}$$

$$\begin{aligned}((123)(45))^3 &= (123)^3(45)^3 \\ &= \underline{(123)^3}\ \underline{(45)^2}(45) \\ &= (45)\end{aligned}$$

$$\begin{aligned}((123)(45))^4 &= (123)^4(45)^4 \\ &= (123)\end{aligned}$$

$\Rightarrow o((123)(45))$ is a common multiple of the length of 123 and the length of 45. By def

$$o\left((123)(45)\right) = \mathrm{lcm}\left(o(123),\ o(45)\right)$$