Reading Questions 12

page 69: Definition 2.79

1. Let A and B be groups. Then $AB = \{ab | a \in A, b \in B\}$.
2. Let A and B be subgroups of G. Then $AB \subseteq G$.
3. Let $A =< 2 >, B =< 3 >$ and $G = Z_6$. Write the elements of BA.

Section 2.6 Subgroups (Part 2)

special subgroups

P 1. Let $X = \{(R_0, R_{180})\}$ and $G = D_8 \times D_8$. Find $C_G(X)$.

P 2. Let G be a group such that $H, K \leq G$. Prove $H \cap K \leq G$.

P 3. Let G and H be groups, and let $\phi : G \to H$ be a homomorphism. The set $\{x \in G \mid \phi(x) = e\}$ is called the kernel of ϕ and is denoted by $\ker(\phi)$. Prove $\ker(\phi) \leq G$.

P 4. Let G be a group, let $H \leq G$ and let $x \in G$. Prove $\{xhx^{-1} \mid h \in H\} \leq G$.

2.6 Let G be a group and $a \in G$. Then $\langle a \rangle \leq G$.

Def: Let X be a set. The subset Y of X is the smallest subset of X that has property P if Y has property P and if $Z \subseteq X$ with property P then $Y \subseteq Z$.

Ex: Let $X = \mathbb{Z}$ and $P := x > 0$. Then X has no smallest subset of positive integers.

$\{1,2,3\} \nsubseteq \{1\} \nsubseteq \{2\}$

Ex: Let $X = [3]$ and $P :=$ contains 1.

Then $\{1\}$ is the smallest subset of X which contain P.

if $1 \in Y \subseteq X$ then $\{1\} \subseteq Y$.

lem: Let G be a group such that $a \in G$. Then $\langle a \rangle$ is the smallest subgroup of G which contains a.

pf: Let H be the smallest subgroup of G which contains a. We know $\langle a \rangle \leq G$ and $a \in \langle a \rangle$.

WTS $\langle a \rangle \leq H$. Since $H \leq G$ and $a \in H$ it follows that $a, a^2, a^3, \ldots \in H$ since H is closed.

Hence $\langle a \rangle \leq H$. $\therefore \langle a \rangle = H$.

Def: Let G be a group and let $X \subseteq G$. Then

$\langle X \rangle :=$ the subgroup generated by X, is the smallest subgroup containing all elements of X.

Ex: Let $X = \{0, 2\}$ and $G = \mathbb{Z}_4$. Then

$\langle X \rangle = \langle 2 \rangle = \{0, 2\}$

Ex: Let $X = \{R_{180}, H\}$ and $G = D_8$. Then

$\langle X \rangle = \{ R_0, R_{180}, H, R_{180}H = HR_{180} = V, VR_{180} = H = R_{180}V = H$

$VH = HV = R_{180} \}$

by constrvction $\langle x \rangle$ is closed

$R_0 \in \langle X \rangle$

by G $\langle x \rangle$ is associative

$\forall y \in \langle x \rangle$, $y \neq e = R_0$ $\quad o(y) = 2 \Rightarrow y^{-1} = y \Rightarrow \langle x \rangle$ has all inverses

$\therefore$ $\langle x \rangle$ is a subgroup of G.

<u>Def:</u> Let G be a group and $X \subseteq G$.

$C_G(X) = \{ x \in G : xa = ax \ \forall a \in X \}$

$:=$ the centaralizer of X

$C_G(G) :=$ the center of G

<u>Ex:</u> Let $X = \{0, 1\}$ and $G = \mathbb{Z}_4$. Then

$C_G(X) = G$ since G is abelian.

$\quad\ \ \shortparallel$

$C_G(G)$

<u>Ex:</u>

Let $X = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix} \right\}$ and $G = GL(2, \mathbb{Z}_3)$.

Then $\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 2 & 2 \\ 1 & 2 \end{bmatrix} \right\} \subseteq C_G(X)$

$$\begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 2 & 2 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 2 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \implies \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 2 & 2 \\ 1 & 2 \end{bmatrix}$$

Thm: Let G be a group such that $X \leq G$. Then

$C_G(X) \leq G$.

pf:

(1) $e_G \in C_G(X)$:= clear $a \in X$ then $e_G a = a e_G = a$

(2) $v, w \in C_G(X)$. Then $va = av$ $\forall a \in X$

and $wa = aw$ $\forall a \in X$.

Hence $vwa = vaw = avw \implies vw \in C_G(X)$

(3) Let $v \in C_G(X)$ WTS $v^{-1} \in C_G(X)$

$va = av$ $\forall a \in X$ $\implies v^{-1}va = v^{-1}av$

$\implies a = v^{-1}av$

$\implies av^{-1} = v^{-1}avv^{-1}$

$\implies av^{-1} = v^{-1}a$.

$\therefore$ $v^{-1} \in C_G(X)$.

(4) associativity follow since G is a group