Reading Questions 11

page 61: Definition 2.60

page 62: Lemma 2.63 and its proof

1. If H is a subgroup of G then the identity element of G is the identity element for H. T
2. If G is a group then the empty set is a subgroup of G. F
3. If G is a group then G is a subgroup of G. T
4. Let $(G, \cdot)$ be a group, and let H be a non-empty subset of G. The subset H is a subgroup of G if $(H, \cdot)$ is a group. Why does H need to be nonempty in this definition?

Section 2.6 Subgroups (Part 1)

subgroups

P 1. List all the subgroups of D_8.

P 2. Prove or disprove: $(\{0, 1, 2, 3, 4, 5\}, +)$ is a subgroup of Z_8.

P 3. Prove the following statement. Let G be a group and let H be a nonempty subset of G. If $ab^{-1} \in H$ for all $a, b \in H$ then H is a subgroup of G.

minimal subgroups

P 4. Let $X = [5]$ and $P :=$ contains only even numbers. Does X have a smallest subset containing P? If so, what is it?

P 5. Let $X = \{R_{180}, D\}$ and $G = D_8$. Find $< X >$.

2.6

Ex: $\langle 2 \rangle \leq \mathbb{Z}_6$ since $\langle 2 \rangle = \{0, 2, 4\} \subseteq \mathbb{Z}_6$ and we $\langle 2 \rangle$ is a group.

Thm: Let G be a group. Then $\forall a \in G$, $\langle a \rangle \leq G$ since $\langle a \rangle$ is a group.

Ex: $\langle 0 \rangle, \underset{\mathbb{Z}_4}{\underset{\shortparallel}{\langle 1 \rangle}}, \langle 2 \rangle$ are all subgroups of $\mathbb{Z}_4$.

This list is complete! If H is a subgroup of $\mathbb{Z}_4$

$0 \in H$. If $1 \in H$ then $H = \langle 1 \rangle = \mathbb{Z}_4$. Similarly if $3 \in H$ then $3^{-1} = 1 \in H$. Hence $\langle 1 \rangle = \langle 3 \rangle = H = \mathbb{Z}_4$. If $2 \in H$ then $H = \langle 2 \rangle$. or $H = \langle 0 \rangle$,

Fact: Let G be group. Then $\langle e_G \rangle, G \leq G$.

Ex: Let $H = \{0, 2, 3\}$. Then H is not a subgroup of $\mathbb{Z}_4$ since $2 + 3 = 1 \notin H$. This shows that H is not closed $\Rightarrow$ H is not a group.

lem: Let G be a group and let $\emptyset \neq H \overset{\text{subset}}{\subseteq} G$. Then $H \overset{\text{subgroup}}{\leq} G$ iff

(1) H is closed

(2) $\forall_{h \in H}\ h^{-1} \in H$.

pf: $(\Rightarrow)$ WTS that H is closed and $\forall_{h \in H}, h^{-1} \in H$,

Assume H is a subgroup of G. Then H is a group. Hence H is closed and $\forall_{h \in H}\ h^{-1} \in H$,

$(\Leftarrow)$ WTS $H \leq G$ or H is a group

(1) identity: $H \neq \emptyset$ so $\exists_{h \in H}$. By hyp. $h^{-1} \in H$. Since is closed $hh^{-1} \in H$. Hence $e_G \in H$.

(2) closed : by hyp

(3) inverse : by hyp

(4) associativity: since G is a group

Thm: Let G be an abelian group. Let

$H = \{ x \in G : x^2 = e_G \}$. Then $H \leq G$.

pf: WTS (1) $hg \in H$ for $h, g \in H$ (2) $\forall h \in H$ $h^{-1} \in H$.

Let $h, g \in H$. Then
$$\begin{aligned} (hg)^2 &= hghg \\ &= hhgg \quad \text{since } G \text{ is abelian} \\ &= h^2 g^2 \\ &= e_G e_G \quad \text{since } h, g \in H \\ &= e_G . \end{aligned}$$

Hence $hg \in H$. Now $h^2 = e_G$ since $h \in H$

$$\begin{aligned} &\Rightarrow hh = e_G \\ &\Rightarrow \underbrace{h^{-1} h}_{}\underbrace{h h^{-1}}_{} = h^{-1} \underbrace{e_G h^{-1}}_{} \\ &\Rightarrow e_G e_G = h^{-1} h^{-1} \\ &\Rightarrow e_G = (h^{-1})^2 . \end{aligned}$$

Therefore $h^{-1} \in H$ and by the previous result $H \leq G$.