Reading Questions 10

page 59: Definition 2.55

$A \times B = \{(a,b) \mid a \in A, b \in B\}$

page 59: Lemma 2.56 and its proof

1. The direct product of two set is a set. T
2. The direct product of two groups is a group. T $(g^{-1}, h^{-1}) = (g,h)^{-1}$
3. Let G and H be groups with identity elements e_G and e_H respectively. What is the identity element in $G \times H$?

   (e_G, e_H)

Section 2.5 Direct Products (Part 1)

Direct Products

P 1. Write the multiplication table for $Z_2 \times Z_2$.

P 2. Let $(1,2) \in Z_2 \times Z_3$. What is $o((1,2))$?

P 3. Find a group that is isomorphic to $Z_2 \times Z_2$.

P 4. Find groups G and H such that $|G| = |H| = 25$ but G is not isomorphic to H.

P 5. Suppose $G \times H$ is abelian. Prove that G and H are abelian.

<u>lem:</u> $G \times H$ is a group.

<u>Ex:</u> $Z_2 \times Z_2 = \{(0,0), (0,1), (1,0), (1,1)\}$ is a group.

<u>Ex:</u> $Z_m \times Z_n$ is a group where $(0,0)$ is the identity.

$(1,1)$ has order 6.

Let $a, c \in Z_m$ and $b, d \in Z_n$.

$((Z_m \times Z_n), (+,+))$

$(a,b) \cdot (c,d) = (a+c, b+d)$

$(a,b) \cdot (0,0) = (a+0, b+0) = (a,b) \Rightarrow (0,0)$ is the true identity.

$m=2 \quad n=3$

$$(1,1)^6 = (1,1)+(1,1)+(1,1)+(1,1)+(1,1)+(1,1)$$

$$= (1+1+1+1+1+1,\ 1+1+1+1+1+1)$$

$$= (6,6) = (\underset{6 \bmod 3}{\overset{6 \bmod 2}{0}}, 0)$$

Thm: Let $(a,b) \in G \times H$. Then $o(a,b) = \text{lcm}(o(a), o(b))$.

pf: Let $m = \text{lcm}(o(a), o(b))$. WTS $(a,b)^m = (e_G, e_H)$.

$$(a,b)^m = (a^m, b^m) \overset{m = \text{lcm}(o(a), o(b))}{=} (a^{o(a)\cdot s}, b^{o(b)\cdot t})$$

$$= ((a^{o(a)})^s, (b^{o(b)})^t)$$

$$= ((e_G)^s, (e_H)^t) = (e_G, e_H).$$

WTS if $(a,b)^k = (e_G, e_H)$ then $k \geq m$.

$(a,b)^k = (e_G, e_H)$. ~~Then $k \mid m$ by previous results.~~

$(a^k, b^k) = (e_G, e_H) \Rightarrow o(a) \mid k$ and $o(b) \mid k$ by previous results.

k is a common multiple of $o(a)$ and $o(b)$.

$\therefore$ $k \geq \text{lcm}(o(a), o(b))$.

Note: $|G \times H| = |G| \cdot |H|$

Ex: Find two groups of order 99 which are not isomorphic.

$(\mathbb{Z}_{99}, +)$

$1 \in \mathbb{Z}_{99}$ $\quad o(1) = 99$

$\therefore$ $\mathbb{Z}_{99}$ is cyclic

$|\mathbb{Z}_3 \times \mathbb{Z}_{33}| = |\mathbb{Z}_3| \cdot |\mathbb{Z}_{33}| = 3 \cdot 33 = 99$

if $\mathbb{Z}_3 \times \mathbb{Z}_{33}$ is cyclic then

$\exists\, (a,b) \in \mathbb{Z}_3 \times \mathbb{Z}_{33}$ s.t $\underset{=}{\overset{m}{o((a,b))}} = 99$

$$o((a,b)) = \text{lcm}\,(o(a), o(b))$$

$$= \text{lcm}\,(1 \text{ or } 3, \ 1 \text{ or } 3 \text{ or } 11 \text{ or } 33)$$

$$\neq 99$$

$\therefore$ $\mathbb{Z}_3 \times \mathbb{Z}_{33}$ is not cyclic

Since $\mathbb{Z}_{99}$ is cyclic and $\mathbb{Z}_3 \times \mathbb{Z}_{33}$ is not cyclic

$\mathbb{Z}_{99} \not\cong \mathbb{Z}_3 \times \mathbb{Z}_{33}$.