Reading Questions 9

page 55: Definition 2.51

page 56: Lemma 2.52 and its proof

1. A homomorphism is a map. T
2. Let a and b be inverses of each other in some group. Then $ab = e$ by the cancellation property for groups. F
3. Let $\phi : ({Z_5}^{\times}, \cdot) \to (Z_4, +)$ be a homomorphism. What is $\phi(1)$? 0

Section 2.4 Isomorphisms (Part 1)

Homomorphisms

P 1. Define a homomorphism from Z_4 to D_8.

P 2. Show that the map $\phi : Z_3 \to (Z_5)^{\times}$ such that $\phi(0) = 1, \phi(1) = 2$ and $\phi(2) = 4$ is not a homomorphism.

P 3. Give an example of an abelian group G, a non abelian group H, and a homomorphism $\phi : G \to H$.

Isomorphisms

P 4. Let $G = Z_4$ and $H = (Z_5)^{\times}$. Show that $G \cong H$.

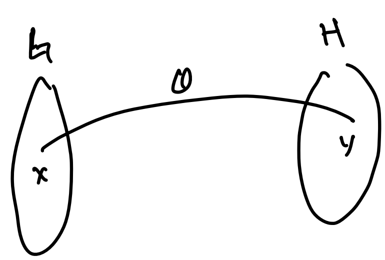
> **Theorem**
>
> Let G and H be groups such that $\phi : G \to H$ is an isomorphism. Then G is abelian if and only if H is abelian. Moreover, $\forall x \in G, o(x) = m$ if and only if $o(\phi(x)) = m$ where m is a positive integer.

P 5. Use the previous theorem to show that D_8 is not isomorphic to Z_8.

P 6. Is Z_6 isomorhic to $(Z_7)^{\times}$?

2.4

map



$\forall x \in G$ $\phi(x)$ exists

Ex: $\phi : [3] \to [5]$ defines a map

$$\phi \left\{ \begin{array}{ccc} 1 & \text{—} & 1 \\ 2 & \text{—} & 2 \\ 3 & \text{—} & 3 \\ & & 4 \\ & & 5 \end{array} \right.$$

or $\phi(1)=1$, $\phi(2)=2$, $\phi(3)=4$

Def: A homomorphism is a map $\theta : G \to H$ such that

$$\theta(ab) = \theta(a)\,\theta(b) \quad \forall a, b \in G$$

Ex: Define θ as

$$\theta \begin{cases} 0 \to 0 \\ 1 \to 1 \\ 2 \to 3 \end{cases} \qquad \begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \qquad \theta : \mathbb{Z}_3 \to \mathbb{Z}_5$$

check

$$\theta(a+b) = \theta(a) + \theta(b)$$

$$\left.\begin{array}{l} \theta(0+1) = \theta(1) = 1 \\ \theta(0) + \theta(1) = 0 + 1 = 1 \end{array}\right\} \therefore \ \theta(0+1) = \theta(0) + \theta(1)$$

$$\vdots$$

$$\theta(0+2) = \theta(0) + \theta(2) = 4 = 0 + 2$$

$$\theta(1+2) = \theta(1) + \theta(2) = 0 = 1 + 4$$

$\therefore$ θ is not a homomorphism.

$$\theta(0+0) = \theta(0) + \theta(0) = 0$$

$$\theta(1+1) = \theta(1) + \theta(1) = 4 \neq 1 + 1$$

Ex:

$(\mathbb{Z}_4, +)$

$$\theta \begin{cases} 0 \to R_0 \\ 1 \to R_{90} \\ 2 \to R_{180} \\ 3 \to R_{270} \end{cases} \qquad \begin{array}{c} H \\ V \\ D \\ D' \end{array}$$

θ is a homomorphism

Def: Let G and H be groups. A map $\phi: G \to H$ is an isomorphism if ϕ is a bijection and a homomorphism. $G \cong H$: G is isomorphic to H.

Ex: Consider $\mathbb{Z}_4$ and $\langle R_{90} \rangle$ where $R_{90} \in D_8$.

$$\phi: \quad
\begin{aligned}
1^0 = 0 &\longmapsto R_0 = (R_{270})^0 \\
1^1 = 1 &\longmapsto R_{270} = (R_{270})^1 \\
1^2 = 2 &\longmapsto R_{180} = (R_{270})^2 \\
1^3 = 3 &\longmapsto R_{90} = (R_{270})^3
\end{aligned}
\qquad \phi(1^i) = (R_{270})^i$$

is an isomorphism. ϕ is clearly a bijection.

$\mathbb{Z}_4 = \langle 1 \rangle$

WTS

$$\phi(1^i + 1^j) = \phi(1^i) + \phi(1^j)$$

$$= \phi(1^{i+j})$$

$$= (R_{270})^{i+j}$$

$$= (R_{270})^i (R_{270})^j = \phi(1^i)\,\phi(1^j)$$

$\phi(1^i + 1^j) = \phi(1^{i+j})$

$= (R_{270})^{i+j}$

Thm: Let G and H be finite cyclic groups such that $|G| = |H|$. Then $G \cong H$.

pf: Let $G = \langle a \rangle$ and $H = \langle b \rangle$. Then $\phi: G \to H$ as $\phi(a^i) = b^i$ is an isomorphism.

$\phi(a^i) = \phi(a^j) \Rightarrow a^i = a^j$ —— 1-1

$\phi(a^i) = b^i$ —— onto

$$\phi(a^i \cdot a^j) = \phi(a^{i+j}) = b^{i+j} = b^i b^j = \phi(b^i)\,\phi(b^j)$$

$\square$

<u>Thm:</u> Let G and H be groups such that $\phi : G \to H$ is an isomorphism.

Then (1) G is abelian $\Leftrightarrow$ H is abelian

(2) $\forall x \in G$ $O(x) = m \Leftrightarrow O(\phi(x)) = m$.

<u>pf:</u> G is abelian then $\forall a, b \in G$ $ab = ba$.

$\phi(ab) = \phi(a)\phi(b)$ since ϕ is a homo.

and $\phi(ba) = \phi(b)\phi(a)$. Since $ab = ba$ $\phi(ab) = \phi(ba)$.

Thus $\phi(a)\phi(b) = \phi(b)\phi(a)$. Let $h_1, h_2 \in H$.

Then $\exists\, g_1, g_2 \in G$ such that $\phi(g_1) = h_1$ and $\phi(g_2) = h_2$ since ϕ is a bijection.

$$h_1 h_2 = \phi(g_1) \cdot \phi(g_2) = \phi(g_1 g_2) = \phi(g_2 g_1) =$$
$$= \phi(g_2)\,\phi(g_1)$$
$$= h_2 h_1$$

$\therefore$ H is abelian.

($\Leftarrow$) left as exercise

(2)

Now let $O(x) = m$. WTS $O(\phi(x)) = m$.

$x^m = e_G$. $(\phi(x))^m = \underbrace{\phi(x) \cdots\cdots \phi(x)}_{m\text{-times}}$

$= \phi(x^m) = \phi(e_G) = e_H$

β $0 < l < m$ and $(\theta(x))^l = e_H$.

Then $e_H = (\theta(x))^l = \underbrace{\theta(x) \cdots \theta(x)}_{l\text{-times}}$

$= \theta(x^l)$.

Since θ is a homo $\theta(e_G) = e_H$. Thus

$\theta(e_G) = \theta(x^l) \Rightarrow e_G = x^l$ by θ is a bijection,

However this contradicts that $o(x) = m$.

($\Leftarrow$) left as an exercise.