Reading Questions 19

$A\vec{v} = \lambda\vec{v}$

$A\,k\vec{v} = \lambda\,k\vec{v}$

Example 7.3.2

1. The distinct nonzero vectors $\vec{v}_1$ and $\vec{v}_2$ both can be eigenvectors of the eigenvalues λ. T
2. If λ is an eigenvalue of A then $\ker A - \lambda I$ can be used to find an eigenvector of A. T
3. What is the eigenspace of matrix? $\ker A - \lambda I$

$A(c_1\vec{v}_1 + c_2\vec{v}_2)$

$= \lambda(c_1\vec{v}_1 + c_2\vec{v}_2)$

Section 7.3 Finding the eigenvectors of a matrix (Part 1)

Eigenspaces

P 1. Find the eigenvectors for the matrix $A = \begin{bmatrix} -3 & 0 & 4 \\ 0 & -1 & 0 \\ -2 & 7 & 3 \end{bmatrix}$.

$A = S\,\mathrm{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n)\,S^{-1}$

P 2. For each eigenvalue λ of a find the algebraic and geometric multiplicity of λ.

P 3. Are the matrices $A = \begin{bmatrix} -3 & 0 & 4 \\ 0 & -1 & 0 \\ -2 & 7 & 3 \end{bmatrix}$ and $B = \begin{bmatrix} 3 & 0 & 0 \\ 2 & 1 & 0 \\ 2 & 7 & 3 \end{bmatrix}$ similar?

P 4. Is the matrix B diagonalizable?

P 5. Suppose the matrices C and D are similar.

1. Show that the matrices $C - \lambda I$ and $D - \lambda I$ are similar.

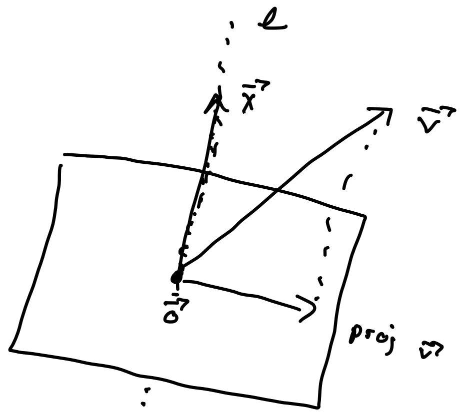
2. What can you conclude about the kernels of the matrices C and D?

3. Show that the geometric multiplicity of C and D are the same.

7.3

Def: Let $A\vec{v} = \lambda \vec{v}$, $\vec{v} \neq \vec{0}$. Then $\ker A - \lambda I$ is the eigenspace of λ, denoted by E_λ.

Ex: Let A be an orthogonal projection on a plane V in $\mathbb{R}^3$. Describe E_1 & E_0.



By (1) $E_1 = V$ and $\dim E_1 = \dim V = 2$

$\uparrow$ geo mult. $\downarrow$

By (2) $E_0 = \ell$ and $\dim E_0 = \dim \ell = 1$

(1) if $\vec{v} \in V$ then $A\vec{v} = \vec{v}$

(2) if $\vec{v} \in V^{\perp} :=$ all vectors perp. to the plane V

$\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ is an eigenbasis where $\{\vec{v}_1, \vec{v}_2\}$ is a basis for V and $\vec{v}_3 \neq \vec{0}$ and on ℓ. In other words let β_1 and β_2 be a basis for E_1 and E_2 respectively.

Then $\beta_1 \cup \beta_2$ is eigenbasis. Therefore the orthogonal projection is diagonalizable.

Thm:

If $A\vec{v}_1 = \lambda_1 \vec{v}_1$ and $A\vec{v}_2 = \lambda_2 \vec{v}_2$ where $\lambda_1 \neq \lambda_2$. Then $\vec{v}_1$ and $\vec{v}_2$ are LI.

pf:

$\underline{\vec{v}_1 = k\vec{v}_2}$ $\Rightarrow$ $A\vec{v}_1 = \lambda_1 \vec{v}_1$ $\vec{v}_1 \in E_1$

$A\, k\vec{v}_2 = \lambda_1 \vec{v}_1$

$$A\vec{v}_2 = A\frac{1}{k}\vec{v}_1$$
$$= \frac{1}{k}\lambda_1 \vec{v}_1$$
$$= \frac{1}{k}\lambda_1 k\vec{v}_2$$
$$= \lambda_1 \vec{v}_2$$

$\Rightarrow \Leftarrow$ since $\lambda_1 \neq \lambda_2$

Ex: Find the eigenvectors of $A = \begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix}$.

First find λ_1, λ_2.

$$A - \lambda I = \begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}$$
$$= \begin{bmatrix} 1-\lambda & 2 \\ 4 & 3-\lambda \end{bmatrix}$$

$$\det A - \lambda I = (1-\lambda)(3-\lambda) - 2 \cdot 4$$
$$= 3 - \lambda - 3\lambda + \lambda^2 - 8$$
$$= \lambda^2 - 4\lambda - 5 = (\lambda - 5)(\lambda + 1)$$

$\Rightarrow$ $\lambda_1 = -1$ & $\lambda_2 = 5$ The eigenvalues are distinct which implies A is diagonalizable,

Next find a basis for E_{-1} and E_5

$E_{-1} = \ker(A - (-1) I)$ $[A + I \;\vdots\; \vec{0}\,]$

$$= \left[\begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \;\vdots\; \vec{0} \right]$$

$$= \left[\begin{array}{cc:c} 2 & 2 & 0 \\ 4 & 4 & 0 \end{array}\right]$$

$$= \begin{bmatrix} 2 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\vec{x} = \begin{bmatrix} -x_2 \\ x_2 \end{bmatrix} = x_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix} \Rightarrow E_{-1} = \text{span}\left(\begin{bmatrix} -1 \\ 1 \end{bmatrix}\right)$$

check $\begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix}\begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix} = -1\begin{bmatrix} -1 \\ 1 \end{bmatrix}$ ✓

$$E_5 = \ker(A - 5I)$$

$$\left[\begin{array}{cc:c} 1-5 & 2 & 0 \\ 4 & 3-5 & 0 \end{array}\right]$$

$$\begin{bmatrix} -4 & 2 & 0 \\ 4 & -2 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -\frac{1}{2} & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\vec{x} = \begin{bmatrix} \frac{1}{2}x_2 \\ x_2 \end{bmatrix} = x_2 \begin{bmatrix} \frac{1}{2} \\ 1 \end{bmatrix} = \hat{x_2} \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$E_2 = \text{span}\left(\begin{bmatrix} 1 \\ 2 \end{bmatrix}\right)$ check $\begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 5 \\ 10 \end{bmatrix} = 5\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ ✓

$\therefore \left\{\begin{bmatrix} -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \end{bmatrix}\right\}$ is an eigenbasis

$\begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix}\begin{bmatrix} -1 & 1 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} -1 & 0 \\ 0 & 5 \end{bmatrix}$ ← check

Def: The geometric multiplicity of the eigenvalue λ is $\dim E_\lambda$.

Thm: Suppose A & B are similar. Then

(1) $\det A - \lambda I = \det B - \lambda I$

(2) A & B have the same eigenvalues