7.1.1
R: 7.1.2

Reading Questions 17

1. If A and B are similar then the matrix A is diagonalizable. F
2. All diagonal matrices are diagonalizable. T
3. If $\{\vec{v}_1, \ldots, \vec{v}_n\}$ is an eigenbasis for the matrix A then the matrix $[\vec{v}_1 \cdots \vec{v}_n]$ is invertible. T
4. If $S^{-1}AS = B$ for some invertible S and diagonal B what can you say about the columns of S?

Section 7.1 Diagonalization (Part 1)

Eigenvalues and Eigenvectors

P 1. Write down what it means for the vector $\vec{v}$ to be an eigenvector of the matrix A.

P 2. If the vector $\vec{v}$ is an eigenvector for the matrix A is $\vec{v}$ an eigenvector for the matrix kA for some nonzero constant k? Explain your answer.

P 3. Does there exist an invertible matrix S and a diagonal matrix B such that $AS = SB$ where A is the linear transformation which rotates a vector 180° in $\mathbb{R}^2$? Explain your answer.

P 4. If $\begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix}$ is an eigenvector of the matrix $\begin{bmatrix} 4 & 1 & 1 \\ -5 & 0 & -3 \\ -1 & -1 & 2 \end{bmatrix}$ what is its eigenvalue?

P 5. Let $\vec{v}$ be an eigenvector for A. Is $\vec{v}$ an eigenvector for A^3? If so what are the eigenvalues?

P 6. Let $\vec{v}$ be an eigenvector for A. Is $\vec{v}$ an eigenvector for $A+\sigma I$? If so what are the eigenvalues?

Recall: Let A be $n \times n$ and $\beta = \{\vec{v}_1, \ldots, \vec{v}_n\}$ and B be the β-matrix. Then $AS = SB$ where

$$S = \begin{bmatrix} | & | & & | \\ \vec{v}_1 & \vec{v}_2 & \cdots & \vec{v}_n \\ | & | & & | \end{bmatrix}$$

Ex:

$$A = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}$$

Find A^5

rank A

det A

a basis for ker A

$$A^5 = \begin{bmatrix} (-1)^5 & 6 & 0 & 0 \\ 0 & 0^5 & 0 & 0 \\ 0 & 0 & 1^5 & 0 \\ 0 & 0 & 0 & 2^5 \end{bmatrix}$$

$\operatorname{rank} A = 3 = 4 - \dim(\ker A)$

$\det A = 0 = (-1)(0)(1)(2)$

$\ker A = \operatorname{span}\left(\begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} \right)$

$$\operatorname{rref}[A \vdots \vec{0}] = \left[\begin{array}{cccc:c} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right]$$

$A\vec{x} = \vec{0}$ $\quad \vec{x} = \begin{bmatrix} 0 \\ x_2 \\ 0 \\ 0 \end{bmatrix} = x_2 \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$

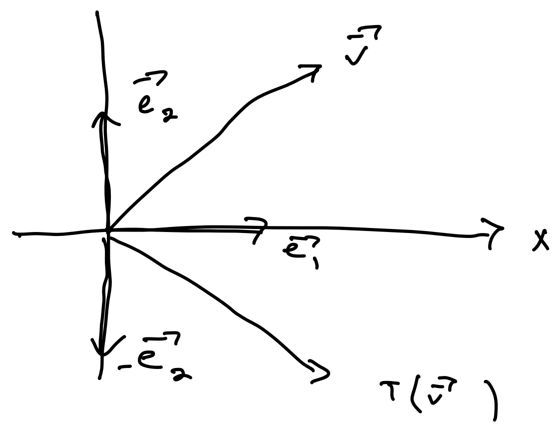
Def: A matrix A is diagonalizable if there exists an invertible S such that $S^{-1}AS = B$ is diagonal.

Ex: The LT T from $\mathbb{R}^2$ to $\mathbb{R}^2$ defined as a reflection about the x-axis is diagonalizable.

$T(\vec{e}_1) = \vec{e}_1$

$T(\vec{e}_2) = -\vec{e}_2$

$\mathcal{B} = \{ \vec{e}_1, \vec{e}_2 \}$ $\quad S = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$



$$B = \begin{bmatrix} | & | \\ [T(\vec{e}_1)]_{\mathcal{B}} & [T(\vec{e}_2)]_{\mathcal{B}} \\ | & | \end{bmatrix}$$

$$= \begin{bmatrix} | & | \\ [\vec{e}_1]_{\mathcal{B}} & [-\vec{e}_2]_{\mathcal{B}} \\ | & | \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = S^{-1}AS$$

Since B is diagonal A is diagonalizable.

Def: $A^{n \times n}$

eigenvector of A

$$A\vec{v} = \lambda \vec{v} \qquad \vec{v} \neq 0$$

eigenvalue of A corresponding to $\vec{v}$

if $\vec{v}_1, \dots, \vec{v}_n$ are eigenvectors of A and linear ind. (a basis for $\mathbb{R}^n$)

then $\{\vec{v}_1, \dots, \vec{v}_n\}$ is an eigenbasis.

Thm: If $\vec{v}$ is an eigenvetor of A then $\vec{v}$ is an eigenvector for A^2.

pf: let $A\vec{v} = \lambda \vec{v}$ then $AA\vec{v} = A\lambda\vec{v}$.

$$\Rightarrow A^2\vec{v} = \lambda \underbrace{A\vec{v}}$$

$$\Rightarrow A^2\vec{v} = \lambda\lambda\vec{v}$$

$$= \lambda^2 \vec{v}$$

eigenval. eigenvec

Thm: The matrix $A^{n \times n}$ is diagonalizable if and only if there exists an eigenbasis for A.

$$AS = SD \quad (D \text{ diag})$$

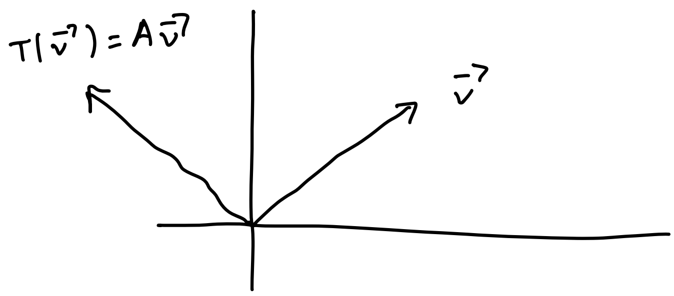
$$A\begin{bmatrix} | & | & & | \\ \vec{v}_1 & \vec{v}_2 & \cdots & \vec{v}_n \\ | & | & & | \end{bmatrix} = \begin{bmatrix} | & | & & | \\ A\vec{v}_1 & A\vec{v}_2 & \cdots & A\vec{v}_n \\ | & | & & | \end{bmatrix}$$

$$= \begin{bmatrix} | & | & & | \\ \lambda_1\vec{v}_1 & \lambda_2\vec{v}_2 & \cdots & \lambda_n\vec{v}_n \\ | & | & & | \end{bmatrix}$$

$$= \begin{bmatrix} | & & | \\ \vec{v}_1 & \cdots & \vec{v}_n \\ | & & | \end{bmatrix} \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix}$$

check

Ex: Let $T(\vec{x}) = A\vec{x} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \vec{x}$. A is a rotation $90°$. Find all eigenvalues and eigenvectors of A.



$T(\vec{v}) \neq \lambda \vec{v}$ for any $\vec{v}$. This means the eigenvectors and eigenvalues are not real.