Reading Questions 12

page 150: example 2

1. The vectors $\begin{bmatrix}4\\1\end{bmatrix}$ and $\begin{bmatrix}-1\\3\end{bmatrix}$ form a basis for $\mathbb{R}^2$. T

   $\{\vec{e}_1, \vec{e}_2\}$ is a basis for $\mathbb{R}^2$

2. If the columns of A form a basis $\mathfrak{B}$ then $[\vec{b}]_{\mathfrak{B}}$ is the solution to $A\vec{x} = \vec{b}$. T

3. What is the vector $[\vec{x}]_{\mathfrak{B}}$ called?

   coordinate vector of x with respect to $\mathcal{B}$

Section 3.4 Coordinates (Part 1)

The coordinate vector

P 1. Write down a way to find the $\mathfrak{B}$-coordinates to a vector $\vec{x}$.

P 2. Find $[\vec{x}]_{\mathfrak{B}}$ where $\mathfrak{B} = \{\begin{bmatrix}1\\2\end{bmatrix}, \begin{bmatrix}5\\6\end{bmatrix}\}$ and $\vec{x} = \begin{bmatrix}-4\\4\end{bmatrix}$.

P 3. Find $[\vec{x}]_{\mathfrak{B}}$ given that $[\vec{v}]_{\mathfrak{B}} = \begin{bmatrix}4\\12\end{bmatrix}$ and $2\vec{v} = \vec{x}$.

P 4. Find the $\mathfrak{B}$-matrix for the linear transformation $T(\vec{x}) = A\vec{x}$, where

$$A = \begin{bmatrix}0 & 1\\2 & 3\end{bmatrix}$$

and $\mathfrak{B} = \{\begin{bmatrix}1\\2\end{bmatrix}, \begin{bmatrix}1\\1\end{bmatrix}\}$.

$\vec{x} = \begin{bmatrix}10\\10\end{bmatrix}$ find $[x]_{\mathcal{B}}$

$$\vec{x} = c_1\vec{v}_1 + c_2\vec{v}_2$$
$$= 4\vec{v}_1 + 2\vec{v}_2$$
$$\Rightarrow [\vec{x}]_{\mathcal{B}} = \begin{bmatrix}4\\2\end{bmatrix}$$

Recall:

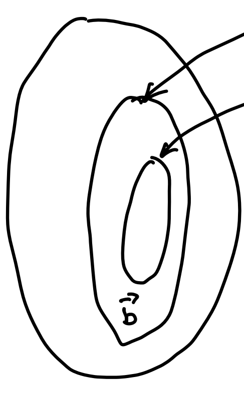
R^n - vector space

W - subspace

basis for $W := \{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_k\} = \mathcal{B}$

or

$:= \{\vec{w}_1, \vec{w}_2, \dots, \vec{w}_k\} = \mathcal{H}$



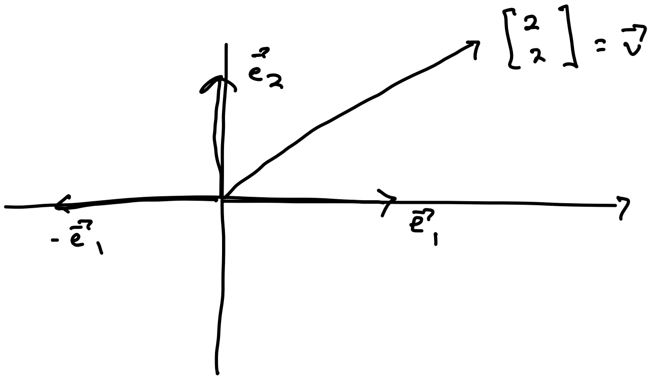
$$W = \text{span}(\vec{v}_1, \dots, \vec{v}_k) = \text{span}(\vec{w}_1, \dots, \vec{w}_k)$$

$$\vec{b} = c_1\vec{v}_1 + c_2\vec{v}_2 + \dots + c_k\vec{v}_k \qquad c_i \in R$$

$$= d_1\vec{w}_1 + d_2\vec{w}_2 + \dots + d_k\vec{w}_k \qquad w_i \in R$$

$$[\vec{b}]_{\mathcal{B}} = \begin{bmatrix}c_1\\c_2\\\vdots\\c_k\end{bmatrix} \qquad [\vec{b}]_{\mathcal{H}} = \begin{bmatrix}d_1\\d_2\\\vdots\\d_k\end{bmatrix}$$

Ex: $\mathcal{B} = \{\vec{e}_1, \vec{e}_2\}$ is a basis for $\mathbb{R}^2$.



$$[\vec{v}]_{\mathcal{B}} = \begin{bmatrix} 2 \\ 2 \end{bmatrix} = \vec{v}$$

$\mathcal{H} = \{-\vec{e}_1, \vec{e}_2\}$

$$[\vec{v}]_{\beta} = \begin{bmatrix} -2 \\ 2 \end{bmatrix} \rightarrow \begin{bmatrix} 2 \\ 2 \end{bmatrix} = -2(-\vec{e}_1) + 2\vec{e}_2$$
$$= -2\begin{bmatrix} -1 \\ 0 \end{bmatrix} + 2\begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Def: The coordinate vector of $\vec{b}$ with respect to the basis $\mathcal{B} = \{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_k\}$ is the vector $\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_k \end{bmatrix} = [\vec{b}]_{\mathcal{B}}$ where $\vec{b} = x_1\vec{v}_1 + \dots + x_k\vec{v}_k$. Hence

$$\underbrace{\begin{bmatrix} | & | & & | \\ \vec{v}_1 & \vec{v}_2 & \cdots & \vec{v}_k \\ | & | & & | \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_k \end{bmatrix}}_{\vec{x}} = \underbrace{\vec{b}}_{\vec{b}},$$

Ex: Let $\mathcal{B} = \left\{\begin{bmatrix} 3 \\ 4 \end{bmatrix}, \begin{bmatrix} 1 \\ 5 \end{bmatrix}\right\}$ and $\mathcal{H} = \left\{\begin{bmatrix} 5 \\ 6 \end{bmatrix}, \begin{bmatrix} -1 \\ -1 \end{bmatrix}\right\}$.

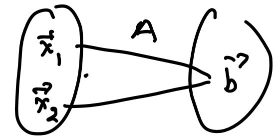
Suppose $[\vec{b}]_{\mathcal{B}} = \begin{bmatrix} 3 \\ -5 \end{bmatrix}$. What is $[\vec{b}]_{\mathcal{H}}$?

$$\begin{bmatrix} 5 & -1 \\ 6 & -1 \end{bmatrix}^{-1} \begin{bmatrix} 4 \\ -13 \end{bmatrix} = [\vec{b}]_{\mathcal{H}}.$$

Thm: $[\vec{b}+\vec{c}]_\beta = [\vec{b}]_\beta + [\vec{c}]_\beta$ and $k[\vec{b}]_\beta = [k\vec{b}]_\beta$

$$A[\vec{b}+\vec{c}]_\beta = \vec{b}+\vec{c}$$

$$A([\vec{b}]_\beta + [\vec{c}]_\beta) = A[\vec{b}]_\beta + A[\vec{c}]_\beta$$
$$= \vec{b}+\vec{c}$$



The columns of A form a basis which implies A^{-1} exists.

$$\therefore \quad A[\vec{b}+\vec{c}]_\beta = A([\vec{b}]_\beta + [\vec{c}]_\beta)$$

$$\Rightarrow \quad A^{-1}A[\vec{b}+\vec{c}]_\beta = A^{-1}A([\vec{b}]_\beta + [\vec{c}]_\beta)$$

$$\Rightarrow \quad [\vec{b}+\vec{c}]_\beta = [\vec{b}]_\beta + [\vec{c}]_\beta .$$

Thm: Let $\mathcal{B} = \{\vec{v}_1, \ldots, \vec{v}_n\}$ be a basis and $T: \mathbb{R}^n \to \mathbb{R}^n$ is a L.T.

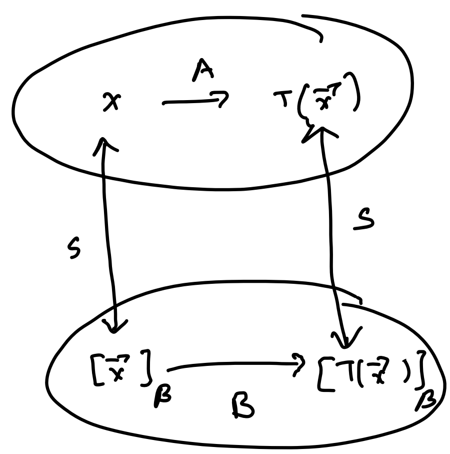
Then there exists a matrix B such that

$$[T(\vec{x})]_\beta = B[\vec{x}]_\beta \text{, called } \mathcal{B}\text{-matrix of } T$$

and

$$B = \begin{bmatrix} | & & | \\ [T(\vec{v}_1)]_\beta & \cdots & [T(\vec{v}_n)]_\beta \\ | & & | \end{bmatrix}.$$

$$\begin{tikzcd}
\vec{x} \arrow[r, "A"] & T(\vec{x}) \\
{[\vec{x}]_\beta} \arrow[u, "S"] \arrow[r, "B"'] & {[T(\vec{x})]_\beta} \arrow[u, "S"']
\end{tikzcd}$$



Ex: Find the $\mathcal{B}$-matrix for the LT $T(\vec{x}) = A\vec{x}$ where

$$A = \begin{bmatrix} 4 & -4 & -2 \\ -4 & 5 & -2 \\ -2 & -2 & 8 \end{bmatrix}$$

$$\mathcal{B} = \left\{ \underset{\vec{v}_1}{\begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix}}, \underset{\vec{v}_2}{\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}}, \underset{\vec{v}_3}{\begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}} \right\}. \quad \Rightarrow \quad S = \begin{bmatrix} 2 & 1 & 0 \\ 2 & -1 & 1 \\ 1 & 0 & 2 \end{bmatrix}$$

$$B = \begin{bmatrix} | & | & | \\ [A\vec{v}_1]_{\mathcal{B}} & [A\vec{v}_2]_{\mathcal{B}} & [A\vec{v}_3]_{\mathcal{B}} \\ | & | & | \end{bmatrix}$$

$$= \left[\begin{bmatrix} -2 \\ 0 \\ 0 \end{bmatrix}_{\mathcal{B}} \begin{bmatrix} 8 \\ 1 \\ 0 \end{bmatrix}_{\mathcal{B}} \begin{bmatrix} -8 \\ -1 \\ 14 \end{bmatrix}_{\mathcal{B}} \right] = \left[S^{-1}\begin{bmatrix} -2 \\ 0 \\ 0 \end{bmatrix} \; S^{-1}\begin{bmatrix} 8 \\ 1 \\ 0 \end{bmatrix} \; S^{-1}\begin{bmatrix} -8 \\ -1 \\ 14 \end{bmatrix} \right]$$

$$= \frac{1}{7}\begin{bmatrix} -4 & 18 & -28 \\ -6 & 20 & 0 \\ 2 & -9 & 63 \end{bmatrix}$$