Reading Questions 11

page 136: example 1

1. If $\vec{v}, \vec{w} \in \mathbb{R}^n$ and the first entry of $\vec{v}$ is nonzero and the first entry of $\vec{w}$ is zero then $\vec{v}$ and $\vec{w}$ are linearly independent. F

2. If $\text{span}(\vec{v}_1, \vec{v}_2, \vec{v}_3) = \ker(A)$ then $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ is a basis for $\ker(A)$. F

3. If $\text{span}(\vec{v}_1, \vec{v}_2) = \ker(A)$ and $\vec{v}_1$ and $\vec{v}_2$ are linearly independent then the dimension of $\ker(A)$ is 2. T

4. Suppose $\text{span}(\begin{bmatrix} 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix}) = V$. What is the dimension of V?

(1) span($\begin{bmatrix} 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}$) and (2) $\begin{bmatrix} 2 \\ 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ are linearly independent

(1) & (2) $\Rightarrow \{\begin{bmatrix} 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}\}$ form a basis for V.

$\therefore$ dim $V = 2$

Section 3.3 The Dimension of a Subspaces in R^n (Part 1)

Dimension

P 1. Let $A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 3 & 6 \end{bmatrix}$.

1. Find a basis for the kernel.
2. Find a basis for the image.
3. Determine the dimensions for each of the previously found subspaces.
4. Use the dimension of the image of A to determine the number of free variables for the system $A\vec{x} = \vec{0}$.
5. Use the dimension of the kernel of A to determine the rank of A.

P 2. For which values of the constant k do the following vectors form a basis for $\mathbb{R}^3$?

$$\begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ k \\ k^2 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$$

P 3. Consider the plane $2x_1 + 3x_2 + x_3 = 0$ which is a subspace of $\mathbb{R}^3$.

1. Find a matrix whose kernel is the same as the plane.
2. Find a basis for the plane.
3. Find the dimension of the plane.

3.2

Ex: Let $A = \begin{bmatrix} 1 & 2 & 1 & 2 \\ 1 & 2 & 2 & 3 \\ 1 & 2 & 3 & 4 \end{bmatrix}$. We know $\text{im } A = \text{span}\left(\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix} \right)$

$$A\vec{x} = \vec{b} = x_1 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + x_2 \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix} + x_3 \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + x_4 \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix}$$

redundant vectors

$$\begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix} = 2 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \Rightarrow \text{im } A = \text{span}\left(\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix} \right)$$

$$\begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \Rightarrow \text{im } A = \text{span}\Big(\underbrace{\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}}_{\text{linearly independent}} \Big)$$

Def: Let $\vec{v}_1, \dots, \vec{v}_m \in \mathbb{R}^n$.

(a) The vector $\vec{v}_i$ is redundant if it is a linear combination of $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_{i-1}$

(b) The vectors $\vec{v}_1, \dots, \vec{v}_m$ are linearly independent if they don't contain a redundant vector. Otherwise the set of vectors are not linearly independent.

(c) If $\vec{v}_1, \dots, \vec{v}_m$ are in a subspace V and linearly independent and they span V ($\Rightarrow V = \text{span}(\vec{v}_1, \dots, \vec{v}_m)$) then they form a basis for V.

Ex: From the previous example $\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix} \right\}$ form a basis for im A. Also $\left\{ \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix} \right\}$ is a basis.

Ex: Let $A = \begin{bmatrix} 1 & 2 & 1 & 2 \\ 1 & 2 & 2 & 3 \\ 1 & 2 & 3 & 4 \end{bmatrix}$. Find a basis for $\operatorname{im}(A)$.

$$\operatorname{rref}[A] = \operatorname{rref}\begin{bmatrix} 1 & 2 & 1 & 2 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 2 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

linearly independent columns of A

$\operatorname{im} A = \operatorname{span}\left(\begin{bmatrix}1\\1\\1\end{bmatrix}, \begin{bmatrix}1\\2\\3\end{bmatrix}\right) \Rightarrow \left\{\begin{bmatrix}1\\1\\1\end{bmatrix}, \begin{bmatrix}1\\2\\3\end{bmatrix}\right\}$ is a basis for $\operatorname{im} A$.

Thm: The set of vectors $\vec{v}_1, \dots, \vec{v}_m$ are linearly independent if all columns of $\operatorname{rref}\begin{bmatrix} | & & | \\ \vec{v}_1 & \cdots & \vec{v}_m \\ | & & | \end{bmatrix}$ contain a pivot.

Ex: The vectors $\begin{bmatrix}1\\2\\3\end{bmatrix}, \begin{bmatrix}4\\5\\6\end{bmatrix}, \begin{bmatrix}7\\8\\9\end{bmatrix}$ are not linearly independent since $\operatorname{rref}\begin{bmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{bmatrix} = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$.

Thm: The vectors $\vec{v}_1, \dots, \vec{v}_m \in \mathbb{R}^n$ are linearly independent if and only if $c_1\vec{v}_1 + \dots + c_m\vec{v}_m = \vec{0}$ implies $c_1 = c_2 = \dots = c_m = 0$

Cor: The vectors $\vec{v}_1, \dots, \vec{v}_m$ form a basis for the subspace V of $\mathbb{R}^n$ if and only if every $\vec{v} \in V$ can be expressed uniquely as $c_1\vec{v}_1 + \dots + c_m\vec{v}_m = \vec{v}$. $c_1, \dots, c_m$ are the coordinates of $\vec{v}$ with respect to $\vec{v}_1, \dots, \vec{v}_m$.

$$\vec{v} = \underline{c_1\vec{v}_1 + \cdots + c_m\vec{v}_m} = \underline{d_1\vec{v}_1 + \cdots + d_m\vec{v}_m} = \vec{v}$$

$$\vec{0} = \vec{v} - \vec{v} = c_1\vec{v}_1 + \cdots + c_m\vec{v}_m - d_1\vec{v}_1 + \cdots + d_m\vec{v}_m$$

$$= (c_1 - d_1)\vec{v}_1 + \cdots + (c_m - d_m)\vec{v}_m$$

3.3

Thm: Let V be a subspaces of $\mathbb{R}^n$. Let $\vec{v}_1, \dots, \vec{v}_p$ be linearly independent in V. Let $\vec{w}_1, \dots, \vec{w}_q$ span V. Then $p \leq q$.

Thm: All bases have the same number of vectors.

Def: The dimension of a subspace V, denoted by $\dim V$, is the number of vectors in a basis for V.

Remarks

$\dim(\operatorname{im} A) = \operatorname{rank} A$

$\dim(\ker A) =$ the # of free variables of $A\vec{x} = \vec{0}$

$= \operatorname{null}(A)$ nullity of A

$\operatorname{rank} A + \operatorname{null} A = n$