# Reading Questions 9

**page 117: example 11**

1. If $T$ is a linear transformation from $\mathbb{R}^n$ to $\mathbb{R}^m$ then the kernel of $T$ is the set of vectors in $\mathbb{R}^n$ which get mapped to the zero vector in $\mathbb{R}^m$. T
2. The kernel of a linear transformation can be found by solving the linear system $A\vec{x} = \vec{1}$. F $A\vec{x} = \vec{0}$
3. What is the kernel of $I_3$? $\vec{0}$

# Section 3.1 The image and kernel of a linear transformation (Part 1)

## The Image

**P 1.** Fill in the blank.

1. The ____________________ of a function $f : X \to Y$ is the set of values the function takes in its target space.
2. If $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_m$ are in $\mathbb{R}^n$. Then the set

$$\{c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_m\vec{v}_m : c_1, \dots, c_m \in \mathbb{R}\}$$

is called ____________________.

**P 2.** Write the image of the linear transformation $\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$.

## The Kernel

**P 3.** Find the vectors that span the image of $\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \end{bmatrix}$.

**P 4.** Write down the definition of the kernel of a linear transformation.

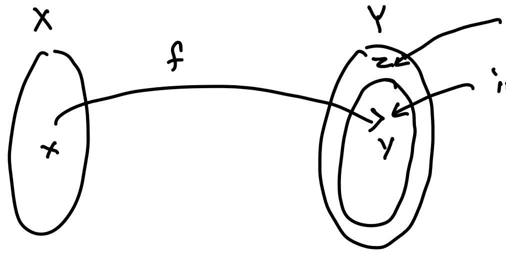
**P 5.** Find a linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^3$ whose image is the line spanned by the vector $\begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix}$.

**P 6.** Find the vectors that span the kernel of $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$.

**P 7.** Assume $A$ is a $n \times m$ matrix. If $\text{rank}(A) = m$ what is the kernel of $A$?

**P 8.** Find a linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^3$ whose kernel is the line spanned by the vector $\begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix}$.

3.1




z is not in im f

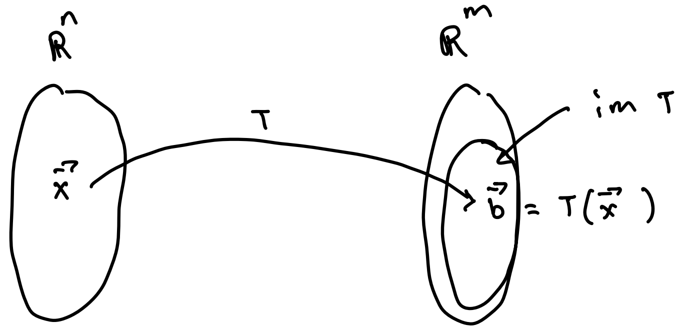
im f := the image of f

$= \{ f(x) : x \in X \}$

$= \{ b \in Y : \exists_{x \in X} \text{ where } f(x) = b \}$

$T(x) = A\vec{x} = \vec{b}$ $\quad x \in \mathbb{R}^n \quad \vec{b} \in \mathbb{R}^m$

$T : \mathbb{R}^n \to \mathbb{R}^m$




<u>Def:</u> The image of a transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ is the set of vectors in its target space that get map to.

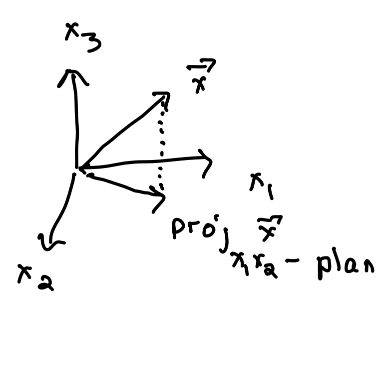
$$\text{im}(T) = \{ T(\vec{x}) : x \in \mathbb{R}^n \} = \{ \vec{y} \in \mathbb{R}^m : \exists_{x \in \mathbb{R}} \text{ where } T(\vec{x}) = \vec{y} \}$$

<u>Ex:</u> If $T : \mathbb{R}^n \to \mathbb{R}^n$ is invertible. then im $T = \mathbb{R}^n$ since for all $\vec{b} \in \mathbb{R}^n$ there exists $\vec{x} \in \mathbb{R}^n$ such that $T(\vec{x}) = \vec{b}$.

$A\vec{x} = \vec{b}$

<u>Ex:</u> Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be LT that projects a vector $\vec{x}$ onto the $x_1 x_2$-plane. Find im T.

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$



If $\vec{v}$ is in $x_1x_2$-plane then $T(\vec{v}) = \vec{v}$. Hence $x_1x_2$-plane is in the im T. All vectors get mapped to the $x_1x_2$-plane which implies im T $= x_1x_2$-plane.

Ex: Let $A = \begin{bmatrix} 1 & 3 \\ 2 & 6 \end{bmatrix}$. Then $T(\vec{x}) = A\vec{x} = \begin{bmatrix} 1 & 3 \\ 2 & 6 \end{bmatrix}\vec{x}$. Find $\operatorname{im}(T)$.

$T: \mathbb{R}^2 \to \mathbb{R}^2$ $\qquad$ $\operatorname{im}(T) = \operatorname{im}(A)$.

$$T(\vec{x}) = \begin{bmatrix} 1 & 3 \\ 2 & 6 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x_1\begin{bmatrix} 1 \\ 2 \end{bmatrix} + x_2\begin{bmatrix} 3 \\ 6 \end{bmatrix} = x_1\begin{bmatrix} 1 \\ 2 \end{bmatrix} + 3x_2\begin{bmatrix} 1 \\ 2 \end{bmatrix}$$
$$= (x_1 + 3x_2)\begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

Hence im T is all vectors parallel to $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$. Also

$$\operatorname{im}(T) = \left\{ k\begin{bmatrix} 1 \\ 2 \end{bmatrix} : k \in \mathbb{R} \right\}$$

Def: Let $v_1, \dots, v_m \in \mathbb{R}^n$. Then $\operatorname{span}(\vec{v}_1, \dots, \vec{v}_m) =$

$\{ c_1\vec{v}_1 + \dots + c_m\vec{v}_m : c_1, c_2, \dots, c_m \in \mathbb{R} \}$. This set is the set of all linear combinations of $\vec{v}_1, \dots, \vec{v}_m$.

Ex: Let $A = \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{bmatrix}$. Then $\operatorname{im} A = \left\{ x_1\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + x_2\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} : x_1, x_2 \in \mathbb{R} \right\}$

$$= \operatorname{span}\left(\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}\right)$$

$$\operatorname{span}\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = \mathbb{R}^2$$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x_1\begin{bmatrix} 1 \\ 0 \end{bmatrix} + x_2\begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

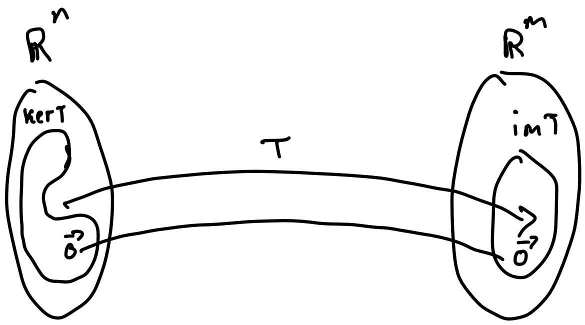
Let $T(\vec{x}) = A\vec{x}$.

**Thm:** $\text{im}(T)$ = the span of the columns of $A$

$= \text{im } A$

**Ex:** In a previous example

$$\text{im } T = \underline{\text{span}\left(\begin{bmatrix}1\\2\end{bmatrix}\right)} = \text{span}\left(\begin{bmatrix}1\\2\end{bmatrix}, \begin{bmatrix}3\\6\end{bmatrix}\right)$$




**Def:** The kernel of a LT $T(\vec{x}) = A\vec{x}$ from $\mathbb{R}^n$ to $\mathbb{R}^m$,

$\ker(T) = \ker(A)$, consists of all solutions to $A\vec{x} = \vec{0}$.

$$\ker(T) = \{\ \vec{x} : \vec{x} \in \mathbb{R}^n \text{ and } T(\vec{x}) = \vec{0}\}$$

**Ex:** Find $\ker(A)$ where $A = \begin{bmatrix}1 & 1 & 1\\1 & 2 & 3\end{bmatrix}$.

$$\left[\begin{array}{ccc|c}1 & 1 & 1 & 0\\1 & 2 & 3 & 0\end{array}\right] \to \left[\begin{array}{ccc|c}1 & 1 & 1 & 0\\0 & 1 & 2 & 0\end{array}\right]$$

$$\to \left[\begin{array}{ccc|c}1 & 0 & -1 & 0\\0 & 1 & 2 & 0\end{array}\right]$$

$$\left.\begin{aligned} x_1 \quad - x_3 &= 0 \\ x_2 + 2x_3 &= 0 \end{aligned}\right\} \qquad \begin{aligned} x_1 &= x_3 \\ x_2 &= -2x_3 \\ x_3 &= x_3 \end{aligned}$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_3 \\ -2x_3 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} \quad \therefore \ \ker A = \operatorname{span}\left( \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} \right)$$

checking:

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$