Reading Questions 8

page 94: example 3

page 95: theorem 2.4.9

1. If $A = \begin{bmatrix} a & 1 \\ 0 & d \end{bmatrix}$ is invertible then a and d are both not 0. T

2. The determinant of $A = \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}$ is 5. F

3. Write down one way to determine if $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is invertible. $\det A \neq 0$ or $\operatorname{rref} A = I_2$ or $\operatorname{rank} A = 2$

Section 2.4 The Inverse of a Linear Transformation (Part 2)

Inverse of linear transformation proofs

P 1. Write down the inverse of the product of two invertible matrices A and B.

If A^{-1} exists then $A^{-1} = \frac{1}{\det A}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$?

P 2. ~~Show that the inverse of a matrix A is unique~~. see notes

By $\operatorname{rref}[A \vdots I_n] = \left[I_n \vdots \frac{1}{\det A}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \right]$

P 3. Find all matrices $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ such that $ad - bc = 1$ and $A^{-1} = A$. Here $ad - bc$ is called the determinant of A.

P 4. Write down the determinant of the following matrices. Explain how they are related.

$$\begin{bmatrix} a & b \\ 0 & d \end{bmatrix}, \begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix}, \begin{bmatrix} a & 0 \\ c & d \end{bmatrix}$$

These matrices are upper triangular, diagonal, and lower triangular matrices.

P 5. Without using row operations, find the inverse of $\begin{bmatrix} 1 & 4 & 4 \\ 0 & 2 & 1 \\ 0 & 2 & 4 \end{bmatrix}$ given that $\begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}^{-1} =$ $\begin{bmatrix} 1 & -2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1/2 \end{bmatrix}$ and $\begin{bmatrix} 1 & 0 & 2 \\ 0 & 2 & 1 \\ 0 & 1 & 2 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 2/3 & -4/3 \\ 0 & 2/3 & -1/3 \\ 0 & -1/3 & 2/3 \end{bmatrix}$.

Q: T or F If $A\vec{x} = \vec{0}$ has a unique solution then A is invertible. T

Thm: Let A and B be invertible $n \times n$ matries. Then

1. $AA^{-1} = A^{-1}A = I_n$

2. $(AB)^{-1} = B^{-1}A^{-1}$

3. A^{-1} is unique

pf: 1) $A^{-1}A\vec{x} = A^{-1}\vec{y} = \vec{x}$ for $\vec{x} \in \mathbb{R}^n$. Hence $A^{-1}A = I_n$.

Similarly $AA^{-1} = I_n$.

2) $AB\,B^{-1}A^{-1} = A\,I\,A^{-1} = AA^{-1} = I$

$B^{-1}A^{-1}AB = B^{-1}I\,B = B^{-1}B = I$

3) Suppose $AC = CA = I$

Then $A^{-1}AC = A^{-1}I$

$\Rightarrow IC = A^{-1}$

$\Rightarrow C = A^{-1}$. $\therefore$ A^{-1} is unique

Thm: Let A and B be $n \times n$ matrices such that $BA = I$. Then A and B are invertible.

Pf: WTS that $A\vec{x} = \vec{b}$ has a unique solution for all $\vec{b}$.

We will show $A\vec{x} = \vec{0}$ has a unique solution.

First, $A\vec{0} = \vec{0}$ so $A\vec{x} = \vec{0}$ has a solution.

Next, $A\vec{y} = \vec{0}$. We will show that $\vec{y} = \vec{0}$.

Then $BA\vec{y} = B\vec{0} = \vec{0} \Rightarrow I\vec{y} = \vec{0} \Rightarrow \vec{y} = \vec{0}$

Therefore $A\vec{x} = \vec{0}$ has a unique solution $\vec{0}$. Moreover

rref $A = I \Rightarrow A$ is invertible.

Since A^{-1} exists we have
$$\begin{aligned} BA = I &\Rightarrow BAA^{-1} = IA^{-1} \\ &\Rightarrow BI = A^{-1} \\ &\Rightarrow B = A^{-1}. \end{aligned}$$

Since A^{-1} is invertible so is B.