Reading Questions 5

page 61-62 Orthogonal Projections

1. The orthogonal projection of a vector onto a line L is a vector that is parallel to L. F
2. The transformation $\text{proj}_L(\vec{x})$ is the orthogonal projection of the vector $\vec{x}$ onto the line L. T
3. Suppose the vector $\vec{u}$ is a unit vector. What is the value of $||\vec{u}||^3$?

   $= 1$

Section 2.2 Linear Transformations in Geometry (Part 1)

Scalings and Orthogonal Projections

P 1. Find the matrix corresponding to the transformation $T(\vec{x}) = 2023\vec{x}$. How does this transformation transform the vector $\vec{x}$?

P 2. Write down the formula and matrix for the orthogonal projection of $\vec{x}$ passing through the line $(0,0)$ and (u_1, u_2) where $u_1^2 + u_2^2 = 1$.

$(\vec{x}\cdot\vec{u})\,\vec{u} = \text{proj}_L(\vec{x}) = \begin{bmatrix} u_1^2 & u_1u_2 \\ u_1u_2 & u_2^2 \end{bmatrix}$

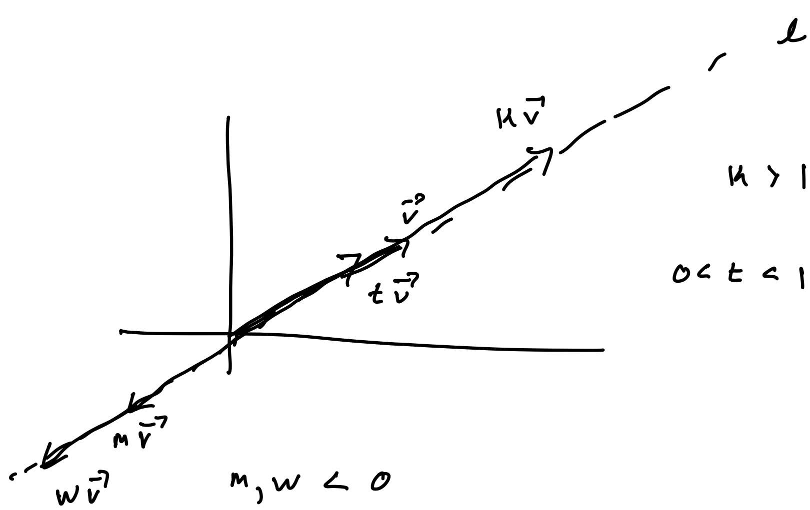
P 3. Let L be the line in $\mathbb{R}^2$ that consists of all scalar multiples of the vector $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$. Find the orthogonal projection of the vector $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ onto L.

Reflections and Rotations

P 4. Write down the matrix for the reflection transformation that reflects vectors over the line L containing the unit vector $\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$.

P 5. Find the matrix for the linear transformation that reflects vectors in $\mathbb{R}^2$ over the line $y = -2x$.

P 6. Find the matrix for the linear transformation that rotates vectors in $\mathbb{R}^2$ by $60°$.



$w < -1$ $\qquad$ $-1 < m < 0$

The formula $\qquad T(\vec{v}) = k\vec{v}$

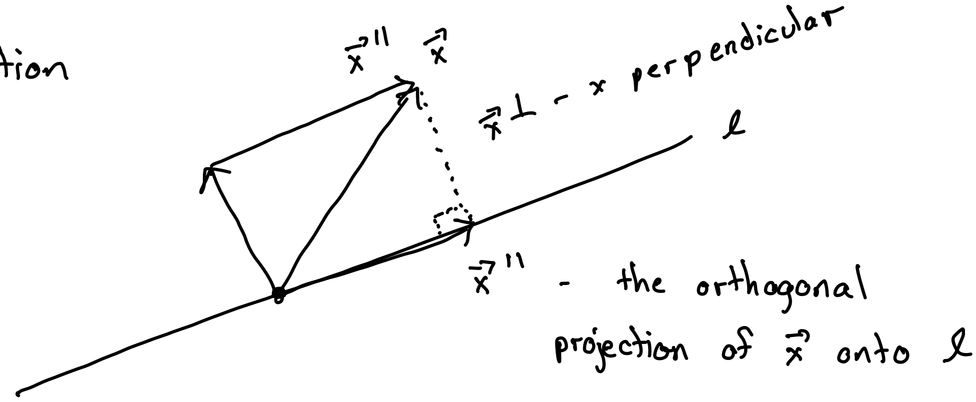
The matrix

$$T\left(\begin{bmatrix}v_1\\v_2\end{bmatrix}\right) = k\begin{bmatrix}v_1\\v_2\end{bmatrix} = k\overbrace{\begin{bmatrix}1&0\\0&1\end{bmatrix}\begin{bmatrix}v_1\\v_2\end{bmatrix}}^{\vec{v}}$$

$$= \begin{bmatrix}k&0\\0&k\end{bmatrix}\begin{bmatrix}v_1\\v_2\end{bmatrix} = A\vec{v}$$

$$\begin{bmatrix}k & & 0\\ & \ddots & \\ 0 & & k\end{bmatrix} \leftarrow \text{Scaling Transformation}$$

Orthogonal Projection



$$\vec{x} = \vec{x}^{\parallel} + \vec{x}^{\perp}$$

$\vec{w}$ is on ℓ

$$\vec{x}^{\parallel} = k\vec{w}$$
$$= \vec{x} - \vec{x}^{\perp}$$

$$\vec{x}^{\perp}\cdot\vec{w} = 0$$

$$(\vec{x} - \vec{x}^{\parallel})\cdot\vec{w} = 0$$

$$(\vec{x} - k\vec{w})\cdot\vec{w} = 0$$

$$\vec{x}\cdot\vec{w} - k\,\vec{w}\cdot\vec{w} = 0$$

$$k = \frac{\vec{x}\cdot\vec{w}}{\vec{w}\cdot\vec{w}}$$

The formula:

$$T(\vec{x}) = \left(\frac{\vec{x}\cdot\vec{w}}{\vec{w}\cdot\vec{w}}\right)\vec{w}$$

$$= \text{proj}_L(\vec{x})$$

if $\vec{w}$ is a unit vector

$$T(\vec{x}) = (\vec{x}\cdot\vec{w})\,\vec{w}$$

The matrix

$$\operatorname{proj}_L\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \left(\frac{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\cdot\begin{bmatrix} w_1 \\ w_2 \end{bmatrix}}{\begin{bmatrix} w_1 \\ w_2 \end{bmatrix}\cdot\begin{bmatrix} w_1 \\ w_2 \end{bmatrix}}\right)\begin{bmatrix} w_1 \\ w_2 \end{bmatrix}$$

$$= \left(\frac{x_1 w_1 + x_2 w_2}{w_1^2 + w_2^2}\right)\begin{bmatrix} w_1 \\ w_2 \end{bmatrix}$$

$$= \frac{1}{w_1^2 + w_2^2}\begin{bmatrix} x_1 w_1 w_1 + x_2 w_2 w_1 \\ x_1 w_1 w_2 + x_2 w_2 w_2 \end{bmatrix}$$

$$= \underbrace{\frac{1}{w_1^2 + w_2^2}\begin{bmatrix} w_1 w_1 & w_2 w_1 \\ w_1 w_2 & w_2 w_2 \end{bmatrix}}_{A}\underbrace{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}}_{\vec{x}}$$

Ex: Consider the line L passing through $(0,0)$ and $(3,4)$.

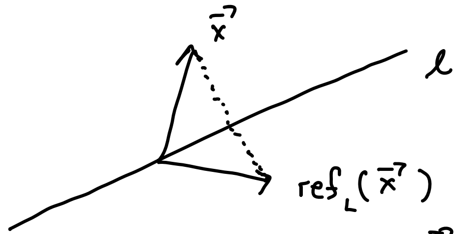
Find $\operatorname{proj}_L(\vec{e}_1)$.

Let $\vec{w} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$

$$\operatorname{proj}_L(\vec{e}_1) = \underbrace{\frac{1}{3^2 + 4^2}\begin{bmatrix} 3^2 & 3\cdot 4 \\ 3\cdot 4 & 4^2 \end{bmatrix}}_{A}\underbrace{\begin{bmatrix} 1 \\ 0 \end{bmatrix}}_{\vec{e}_1}$$

$$= \frac{1}{9+16}\begin{bmatrix} 9 \\ 12 \end{bmatrix}$$

Reflection



$ref_L(\vec{x})$ - the reflection of $\vec{x}$ across ℓ

The formula

$$ref_L(\vec{x}) = 2\, proj_L(\vec{x}) - \vec{x}$$

The matrix

$$ref_L(\vec{x}) = 2\, proj_L(\vec{x}) - \vec{x}$$

$$= 2P\vec{x} - \vec{x} = \underbrace{(2P - I)}_{A}\underbrace{\vec{x}}_{\vec{x}}$$

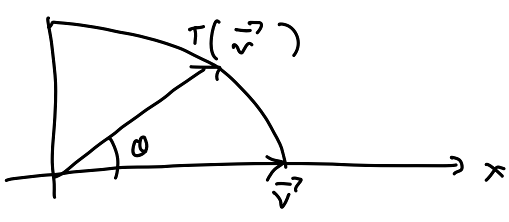
Ex: consider the line L passing through (0,0) and (3,4).

Find $ref_L(\vec{e}_1)$.

$$ref_L(\vec{e}_1) = 2\, proj_L(\vec{e}_1) - \vec{e}_1$$

$$= \frac{2}{9+16}\begin{bmatrix} 3^2 & 4\cdot 3 \\ 3\cdot 4 & 4^2 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} - \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$= \frac{1}{25}\begin{bmatrix} -7 \\ 12 \end{bmatrix}$$ $\leftarrow$ check

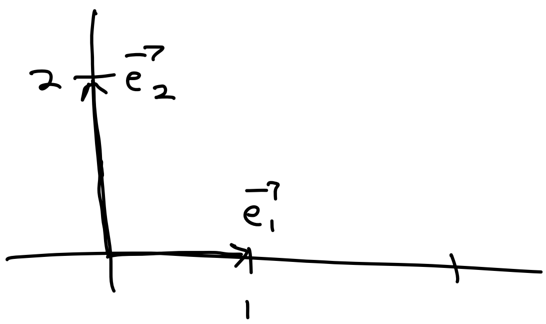


The formula $$T_\theta(\vec{x}) = \cos(\theta)\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \sin(\theta)\begin{bmatrix} -x_2 \\ x_1 \end{bmatrix}$$

The matrix $$T_\theta(\vec{x}) = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}\vec{x}$$

Ex: Find a transformation that maps $\vec{e}_1$ to $2\vec{e}_2$

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \begin{bmatrix} 0 \\ 2 \end{bmatrix}$$



$$\underbrace{\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} \cos 90^\circ & -\sin 90^\circ \\ \sin 90^\circ & \cos 90^\circ \end{bmatrix}}\begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$2\,I_2\begin{bmatrix} \cos 90^\circ & -\sin 90^\circ \\ \sin 90^\circ & \cos 90^\circ \end{bmatrix}$$

$$\begin{bmatrix} 2\cos 90^\circ & -2\sin 90^\circ \\ 2\sin 90^\circ & 2\cos 90^\circ \end{bmatrix}$$