Reading Questions 3

page 26: example 2

page 28: definition 1.3.5 # of pivots

1. The notation rank(A) represents the number of nonzero entries in the rref(A). F
2. The sum of an $n \times n$ matrix A and $n \times n$ matrix B is an $n \times n$ matrix. T
3. List the entries of b if $b = \begin{bmatrix} 2 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ 2 \end{bmatrix}$. $= \begin{bmatrix} 3 \\ 3 \end{bmatrix} = \begin{bmatrix} 2+1 \\ 1+2 \end{bmatrix}$

Section 1.3 On the Solutions of Linear Systems (Part 1)

The rank of a matrix

P 1. For each of the following augmented matrices write its solutions and state the number of solution it has.

$$A = \begin{bmatrix} 1 & 0 & \mid & 5 \\ 0 & 1 & \mid & 6 \end{bmatrix} \quad B = \begin{bmatrix} 1 & 0 & 2 & 1 & \mid & 0 \\ 0 & 1 & 3 & 1 & \mid & 0 \\ 0 & 1 & 3 & 1 & \mid & 0 \\ 0 & 1 & 3 & 1 & \mid & 0 \end{bmatrix} \quad C = \begin{bmatrix} 0 & 1 & 0 & \mid & 2 \\ 0 & 0 & 1 & \mid & 3 \end{bmatrix}$$

P 2. For each of the following matrices write the rref and determine its rank.

$$A = \begin{bmatrix} 1 & 2 & 2 \\ 3 & 2 & 3 \\ 0 & 0 & 0 \end{bmatrix} \quad B = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix} \quad C = \begin{bmatrix} 3 & 3 & 3 \\ 3 & 3 & 3 \\ 3 & 3 & 3 \end{bmatrix}$$

P 3. Suppose that A is an $n \times n$ coefficient matrix and the rank of A is n. How many solutions does the system of equations have? Justify your answer.

Matrix Algebra

P 4. Compute

$$\begin{bmatrix} 1 & 2 & 2 \\ 3 & 2 & 3 \\ 3 & 1 & 0 \end{bmatrix} + \begin{bmatrix} 1 & 2 & 2 \\ 3 & 2 & 3 \\ 2 & 3 & 2 \end{bmatrix}, \quad 4 \begin{bmatrix} 1 & 2 & 2 \\ 3 & 2 & 3 \\ 2 & 3 & 2 \end{bmatrix}, \quad \begin{bmatrix} 1 & 2 & 2 \\ 3 & 2 & 3 \\ 3 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix}.$$

P 5. Compute the product $A\vec{x}$ by using the rows of A.

$$A = \begin{bmatrix} 4 & 4 & 2 \\ 5 & 5 & 1 \\ 3 & 3 & 1 \end{bmatrix} \qquad \vec{x} = \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix}$$

P 6. Let $A = \begin{bmatrix} 1 & 3 & 2 \\ 2 & 3 & 2 \\ 1 & 3 & 1 \end{bmatrix}$ and $\vec{x} = \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix}$. Write $\begin{bmatrix} 13 \\ 15 \\ 12 \end{bmatrix}$ as a linear combination of the columns of A.

1.3

$$\left[\begin{array}{cccc|c} 0 & 0 & 0 & 0 & * \end{array}\right] \to 0 \text{ sol}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & * \\ 0 & 1 & 0 & * \\ 0 & 0 & 0 & 0 \end{array}\right] \to \infty \text{ sols}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & * \\ 0 & 1 & 0 & * \\ 0 & 0 & 1 & * \end{array}\right] \to 1 \text{ sol}$$

Ex:

$$A = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad \text{rank } A = 2$$

$$B = \begin{bmatrix} \boxed{1} & 1 & 1 \\ 0 & \boxed{1} & 1 \\ 0 & 0 & \boxed{1} \end{bmatrix} \qquad \text{rank } B = 3$$

Thm: Let

identity matrix

$$A = I_n = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \cdots & 0 & 1 \end{bmatrix}^{n \times n}$$

rows, cols

A, B - $n \times n$

$$\text{rref } B = I_n \iff \text{rank } B = n$$

Ex:

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} + \begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix} = \begin{bmatrix} 1+5 & 2+6 \\ 3+7 & 4+8 \end{bmatrix} = \begin{bmatrix} 6 & 8 \\ 10 & 12 \end{bmatrix}$$

$$5 \begin{bmatrix} 1 & 2 \\ 3 & 5 \end{bmatrix} = \begin{bmatrix} 1 \cdot 5 & 2 \cdot 5 \\ 3 \cdot 5 & 5 \cdot 5 \end{bmatrix} = \begin{bmatrix} 5 & 10 \\ 15 & 25 \end{bmatrix}$$

Let $A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 0 & -1 \end{bmatrix}$ $\qquad \vec{x} = \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix}$

$$A\vec{x} = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 0 & -1 \end{bmatrix} \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix}$$

$$= \begin{bmatrix} [1\ 2\ 3]\cdot\begin{bmatrix}3\\1\\2\end{bmatrix} \\ [1\ 0\ -1]\begin{bmatrix}3\\1\\2\end{bmatrix} \end{bmatrix} = \begin{bmatrix} 1\cdot 3 + 2\cdot 1 + 3\cdot 2 \\ 1\cdot 3 + 0\cdot 1 + (-1)\cdot 2 \end{bmatrix}$$

$$= \begin{bmatrix} 11 \\ 1 \end{bmatrix}$$

<u>Def:</u> A vector $\vec{b}$ in $\mathbb{R}^n$ is a linear combination of the vectors $\vec{v}_1, \ldots, \vec{v}_m$ in $\mathbb{R}^n$ if there exists $x_1, x_2, \ldots, x_m$ in $\mathbb{R}$ such that

$$\vec{b} = x_1 \vec{v}_1 + \cdots + x_m \vec{v}_m .$$

<u>Ex:</u> we saw

$$A\vec{x} = \begin{bmatrix} 11 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 0 & -1 \end{bmatrix}\begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix}$$

$$= \begin{bmatrix} 1\cdot 3 + 2\cdot 1 + 3\cdot 2 \\ 1\cdot 3 + 0\cdot 1 + (-1)\cdot 2 \end{bmatrix}$$

$$= \begin{bmatrix} 1\cdot 3 \\ 1\cdot 3 \end{bmatrix} + \begin{bmatrix} 2\cdot 1 \\ 0\cdot 1 \end{bmatrix} + \begin{bmatrix} 3\cdot 2 \\ (-1)\cdot 2 \end{bmatrix}$$

$$= 3\begin{bmatrix} 1 \\ 1 \end{bmatrix} + 1\begin{bmatrix} 2 \\ 0 \end{bmatrix} + 2\begin{bmatrix} 3 \\ -1 \end{bmatrix}$$

$\begin{bmatrix} 11 \\ 1 \end{bmatrix}$ is a linear combination of the vectors

$$\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ -1 \end{bmatrix}$$