Reading Questions

page 110: Definition 5.1

1. Let G be a group such that $x \in G$ and $H \subseteq G$. If Hx is a right cosets then H must be a subgroup of G. T
2. Let G be a group such that $x \in G$ and $H \leq G$. Then Hx is a subgroup of G. F
3. Let $G = \mathbb{Z}_5$ and $H =< 2 >$ and $x = 3$. List the elements of Hx. $= \mathbb{Z}_5$

$H = \mathbb{Z}_5 \Rightarrow Hx = \mathbb{Z}_5$

Section 5.1 Translation Action and Cosets (Part 1)

Cosets

P 1. Let $G = S_4$ and $H =< (123) >$. List the right cosets of H in G.

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P 2. Let $G = S_4$ and $H =< (123) >$. What is $|H : G|$?

P 3. Let $G = D_8$ and $H =< R_{90} >$. List the left cosets of H in G.

5.1 Translation Actions and Cosets

Def: Let G be a group such that $H \leq G$ and $x \in G$.

$Hx := \{ hx : h \in H \}$ - Right cosets of H in G

$xH := \{ xh : h \in H \}$ - Left cosets of H in G

Ex: Let $G = D_8$ and $H = R_{180}$. Then

$HR_{90} = \{ R_0 R_{90}, R_{180} R_{90} \} = \{ R_{90}, R_{270} \}$.

$= R_{90} H$

Note: HR_{90} is not a subgroup of G as $R_0 \notin HR_{90}$.

Cor: The distinct right cosets partition the group.

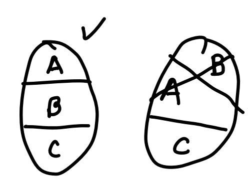
For ex

$Hx = Hy \quad x \neq y$

pf: The right cosets are orbits.

H acts on G

$h \cdot g = hg$



Ex: Let $G = S_3$ and $H = \langle (12) \rangle$. Then

$H(1) = H = H(12) = \{(1), (12)\} = O_G((12))$ H acts on G $h \cdot g = hg$

$H(13) = \{(1)(13), (12)(13)\} = \{(13), (132)\} = H(132)$

$H(23) = \{(1)(23), (12)(23)\} = \{(23), (123)\} = H(123)$

lem: Let G be a group such that $H \leq G$ and $x, y \in G$,

(1) $Hx = Hy \iff y \in Hx$

pf: ($\Rightarrow$) Suppose $Hx = Hy$. Since $H \leq G$, $e \in H$. Hence $y = ey \in Hy$. Since $Hx = Hy$, $y \in Hx$.

($\Leftarrow$) Suppose $y \in Hx$.

($Hx \subseteq Hy$)
Let $z \in Hx$. Then $\exists\, h_1, h_2 \in H$ s.t. $z = h_1 x$ and $y = h_2 x$. Hence $x = h_2^{-1} y \Rightarrow z = \underbrace{h_1 h_2^{-1}}_{\in H} y$.

Thus $z \in Hy$.

($Hy \subseteq Hx$)
Now let $z \in Hy$. Hence $z = h_3 y$, $h_3 \in H$.
Then $z = \underbrace{h_3 h_2}_{\in H} x \Rightarrow z \in Hx$. $\therefore$ $Hx = Hy$.

lem: $Hg = H \iff g \in H$

Def: Let G be a group such that $H \leq G$. The set of all distinct right cosets is G/H. The number of distinct right cosets is $|G/H|$ or $|G:H|$ is the index of H in G.

$\mathbb{Z}/\mathbb{Z}_5 = \mathbb{Z}_5$

Ex: From the previous example, $|S_3 : \langle 12 \rangle| = 3$

lem: Let G be a group such that $H \leq G$. Then

(1) $|G:\{e\}| = |G|$

(2) $|G:G| = 1$

(3) $|G:H| = \#$ of $\underbrace{\text{left cosets of } H}_{xH}$