# Reading Questions

**Example 7.4.2**

1. The limit of a matrix is the limit of its entries. F
2. If the $n \times n$ matrix $A$ is diagonalizable then $A^t$ is a diagonal matrix for some positive integer $t$. F
3. I have used Cocalc on at least one of my homework assignments. T
4. Do you have any questions about Cocalc? If so, what are your questions?

# Section 7.4 More on Dynamical Systems (Part 1)

## Definitions

**P 1.** Give two different examples of a distribution vector.

**P 2.** Give an example of a positive transition matrix.

**P 3.** Suppose $\vec{x}, \vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n$ are distribution vectors. Let the columns of $A$ are the vectors $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n$. Is $A\vec{x}$ a distribution vector. Explain your answer.

**P 4.** What can you say about the columns of the matrix $A^t$ where $A$ is the matrix from the previous problem and $t$ is a positive integer?

## Dynamical System

**P 5.** Let $A = \begin{bmatrix} 0.8 & 0.6 \\ 0.2 & 0.4 \end{bmatrix}$.

1. Use the example in today's lecture to find the closed formula for $A^t$ where $t$ is an arbitrary positive integer.
2. Find $\lim_{t\to\infty} A^t$.
3. Compute $\lim_{t\to\infty} A^t \begin{bmatrix} 0.5 \\ 0.5 \end{bmatrix}$.

Sec 7.4 Dynamical Systems $\underline{x}^t i + {x^2}^{t^2} j \Rightarrow \begin{bmatrix} t \\ t^2 \end{bmatrix}$

The system $\vec{x}(t+1) = A\vec{x}(t)$ is called

a dynamical.

**Ex:** Coyotes & Road runners.

$$\underbrace{\begin{bmatrix} c(t+1) \\ r(t+1) \end{bmatrix}}_{\vec{x}(t+1)} = \underbrace{\begin{bmatrix} 0.86 & 0.08 \\ -0.12 & 1.14 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} c(t) \\ r(t) \end{bmatrix}}_{\vec{x}(t)}$$

where $c(t)$, $r(t)$ is the population of coyotes and roadrunners at time $t$.

If $\vec{x}(t+1) = A\vec{x}(t)$ for $t = 0, 1, 2, 3, \ldots$

then

$$\vec{x}(t+1) = A \underbrace{A\vec{x}(t-1)}_{\vec{x}(t)}$$

$$= AAA\,\vec{x}(t-2)$$

$$\vdots$$

$$= A^t \vec{x}(0) = A^t \vec{x}_0$$

**Def:** A vector $\vec{x}$ in $\mathbb{R}^n$ is a distribution vector if its components add up to 1 and they are nonnegative.

Ex: $\begin{bmatrix} 0.5 \\ 0.5 \end{bmatrix}$, $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $\begin{bmatrix} 1/3 \\ 1/3 \\ 1/3 \end{bmatrix}$ are distribution vectors.

Def: A matrix A is a transition matrix if its columns are distribution vectors.

Ex: $\begin{bmatrix} 0.5 & 1 \\ 0.5 & 0 \end{bmatrix}$, $\begin{bmatrix} 1/3 & 1/3 & 1/3 \\ 1/3 & 1/3 & 1/3 \\ 1/3 & 1/3 & 1/3 \end{bmatrix}$ are transition matrices.

Def: A transition matrix is positive if all its entries are positive.

Ex: $\begin{bmatrix} 0.5 & 1 \\ 0.5 & 0 \end{bmatrix}$ is not positive

$\begin{bmatrix} 0.5 & 0.2 \\ 0.5 & 0.8 \end{bmatrix}$ is positive

Def: A transition matrix is regular if there exists a positive m such that $A^m$ is positive.

Ex: $\begin{bmatrix} 0.5 & 1 \\ 0.5 & 0 \end{bmatrix}^2 = \begin{bmatrix} (0.5)^2+0.5 & 0.5 \\ (0.5)^2 & 0.5 \end{bmatrix}$ so $\begin{bmatrix} 0.5 & 1 \\ 0.5 & 0 \end{bmatrix}$ is regular.

Thm: Let A be a regular transition matrix of size n×n. Then there exists exactly one distribution vector $\vec{x}$ in $\mathbb{R}^n$ such that $A\vec{x} = \vec{x}$.

$\vec{x} :=$ equilibrium distribution vector

$= \vec{x}_{equ}$

$\vec{x}(t+1) = A^t \vec{x}_0$

(2) For all $\vec{x}_0$ distribution vectors

$$\lim_{t \to \infty} A^t \vec{x}_0 = \vec{x}_{equ}$$

(3) $$\lim_{t \to \infty} A^t = \begin{bmatrix} | & | & & | \\ \vec{x}_{equ} & \vec{x}_{equ} & \cdots & \vec{x}_{equ} \\ | & | & & | \end{bmatrix}$$

<u>Recall:</u> If $S^{-1} A S = B = \begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{bmatrix}$

then $S^{-1} A^t S = B^t = \begin{bmatrix} \lambda_1^t & & & \\ & \lambda_2^t & & \\ & & \ddots & \\ & & & \lambda_n^t \end{bmatrix}$

<u>Ex:</u> $$\lim_{t \to \infty} \begin{bmatrix} 1 + (0.01)^t & 4 \\ 0 & \frac{3}{5^t} \end{bmatrix} = \begin{bmatrix} \lim_{t\to\infty} 1 + (0.01)^t & \lim_{t\to\infty} 4 \\ 0 & \lim_{t\to\infty} \frac{3}{5^t} \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 4 \\ 0 & 0 \end{bmatrix}$$

<u>Ex:</u> Consider the matrix $A = \begin{bmatrix} 0.4 & 0.3 \\ 0.6 & 0.7 \end{bmatrix}$

(a) Use diagonalization to find a closed formula for $A^t$. Compute $\lim_{t \to \infty} A^t$.

(b) Find $\lim_{t \to \infty} A^t \begin{bmatrix} 0.5 \\ 0.5 \end{bmatrix}$

$$A = \begin{bmatrix} 0.4 & 0.3 \\ 0.6 & 0.7 \end{bmatrix} = \frac{1}{10}\begin{bmatrix} 4 & 3 \\ 6 & 7 \end{bmatrix} = \frac{1}{10} A'$$

$$A' \vec{x} = \lambda \vec{x}$$
$$10 A \vec{x} = \lambda \vec{x}$$
$$A \vec{x} = \frac{\lambda}{10} \vec{x}$$

$\vec{x}_{equ} = ?$

$= \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

For $A'$ $\lambda_1 = 1$ $\lambda_2 = 10$

$\vec{x}_{\lambda_1} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ $\vec{x}_{\lambda_2} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

$$\begin{bmatrix} 4 & 3 \\ 6 & 7 \end{bmatrix}\begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix} = 1\begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$\lim_{t \to \infty} A^t \begin{bmatrix} 0.5 \\ 0.5 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$\lim_{t \to \infty} A^t = \begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}$$

$$A^t = \begin{bmatrix} 1 & 1 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{10} \end{bmatrix}^t \begin{bmatrix} 1 & 1 \\ -1 & 2 \end{bmatrix}^{-1}$$

$$= \begin{bmatrix} 1 & 1 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{10^t} \end{bmatrix}\begin{bmatrix} 1 & 1 \\ -1 & 2 \end{bmatrix}^{-1}$$