5.2 Gram-Schmidt and QR Factorization

__Def:__ Vectors $\vec{u}_1, \vec{u}_2, \dots, \vec{u}_m$ are orthonormal if

$$\vec{u}_i \cdot \vec{u}_j = 0 \quad \text{and} \quad \|\vec{u}_i\| = 1.$$

__Ex:__ $\vec{e}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \vec{e}_2$ are orthonormal as

$$\vec{e}_1 \cdot \vec{e}_2 = 1 \cdot 0 + 0 \cdot 1 = 0 \quad \text{and} \quad \|\vec{e}_1\| = \sqrt{1^2 + 0^2} = 1$$

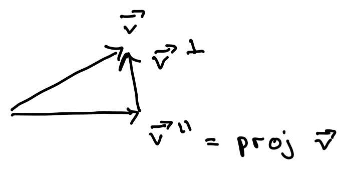
$$\|\vec{e}_2\| = \sqrt{0^2 + 1^2} = 1.$$

__P__ Show that $\begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}$ and $\begin{bmatrix} -\sin\theta \\ \cos\theta \end{bmatrix}$ are orthonormal vectors.

__Thm:__ Let $\vec{u}_1, \vec{u}_2, \dots, \vec{u}_m$ be orthonormal vectors and a basis for $\mathbb{R}^n$. Then

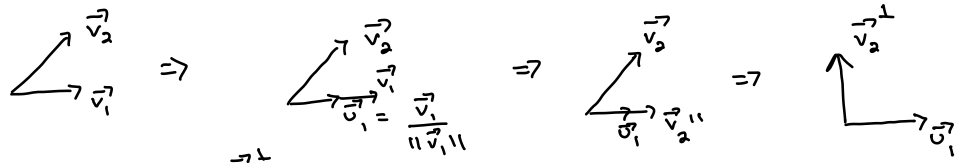
$$\text{proj}_V \vec{x} = \vec{x}^{\parallel} = (\vec{u}_1 \cdot \vec{x})\vec{u}_1 + \dots + (\vec{u}_m \cdot \vec{x})\vec{u}_m.$$

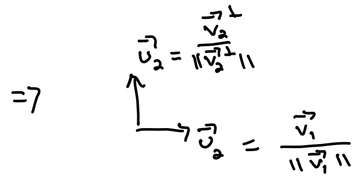
__Recall:__



and $\vec{v} = \vec{v}^{\parallel} + v^{\perp}$

So $\vec{v}^{\perp} = \vec{v} - \vec{v}^{\parallel} = \vec{v} - \text{proj}\,\vec{v}$





Hence $\dfrac{\vec{v}_1}{\|\vec{v}_1\|}$ and $\dfrac{\vec{v}_2^{\,\perp}}{\|\vec{v}_2\|}$ are orthonormal vectors.

Ex: Find orthonormal vectors $\vec{u}_1$ and $\vec{u}_2$ in the subspace $V = \text{span}\left(\begin{bmatrix}1\\1\\1\\1\end{bmatrix}, \begin{bmatrix}1\\9\\9\\1\end{bmatrix}\right)$.

Let $\vec{v}_1 = \begin{bmatrix}1\\1\\1\\1\end{bmatrix}$ and $\vec{v}_2 = \begin{bmatrix}1\\9\\9\\1\end{bmatrix}$

We want to find $\dfrac{\vec{v}_1}{\|\vec{v}_1\|}$ and $\dfrac{\vec{v}_2^{\,\perp}}{\|\vec{v}_2^{\,\perp}\|}$.

$$\vec{u}_1 = \frac{\vec{v}_1}{\|\vec{v}_1\|} = \frac{1}{\sqrt{1^2+1^2+1^2+1^2}}\begin{bmatrix}1\\1\\1\\1\end{bmatrix} = \frac{1}{2}\begin{bmatrix}1\\1\\1\\1\end{bmatrix}$$

$$\vec{u}_2 = \frac{\vec{v}_2^{\,\perp}}{\|\vec{v}_2^{\,\perp}\|}, \quad \vec{v}_2^{\,\perp} = \vec{v}_2 - \text{proj}_V \vec{v}_2$$

$$= \vec{v}_2 - (\vec{u}_1 \cdot \vec{v}_2)\vec{u}_1 \leftarrow \text{By previous theorem}$$

$$\vec{u}_1 \cdot \vec{v}_2 = \frac{1}{2}\begin{bmatrix}1\\1\\1\\1\end{bmatrix} \cdot \begin{bmatrix}1\\9\\9\\1\end{bmatrix} = \frac{1}{2}(1+9+9+1) = 10$$

$$(\vec{u}_1 \cdot \vec{v}_2)\vec{u}_1 = \frac{10}{2}\begin{bmatrix}1\\1\\1\\1\end{bmatrix}$$

$$\|\vec{v}_2^{\,\perp}\| = \sqrt{4^2((-1)^2+1^2+1^2+(-1)^2)} = 4 \cdot 2 = 8$$

$$\vec{v}_2^{\,\perp} = \begin{bmatrix}1\\9\\9\\1\end{bmatrix} - \begin{bmatrix}5\\5\\5\\5\end{bmatrix} = \begin{bmatrix}-4\\4\\4\\-4\end{bmatrix}$$

$$\vec{u}_2 = \frac{1}{8}\begin{bmatrix}-4\\4\\4\\-4\end{bmatrix} = \frac{1}{2}\begin{bmatrix}-1\\1\\1\\-1\end{bmatrix}$$

Check: $\vec{v}_1 \cdot \vec{v}_1 = \vec{v}_2 \cdot \vec{v}_2 = 0$ and $\vec{v}_1 \cdot \vec{v}_2 = 1$

Ex: Find $R^{2\times 2}$ such that $\begin{bmatrix} 1 & 1 \\ 1 & 9 \\ 1 & 9 \\ 1 & 1 \end{bmatrix} = \underbrace{\begin{bmatrix} 1/2 & -1/2 \\ 1/2 & 1/2 \\ 1/2 & 1/2 \\ 1/2 & -1/2 \end{bmatrix}}_{Q} R$

Solve

$$\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 1 & -1 \\ 1 & 1 \\ 1 & 1 \\ 1 & -1 \end{bmatrix} \vec{r}_1 \quad \text{and} \quad \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 1 & -1 \\ 1 & 1 \\ 1 & 1 \\ 1 & -1 \end{bmatrix} \vec{r}_2$$

$$\left[\begin{array}{c:cc} 2 & 1 & -1 \\ 2 & 1 & 1 \\ 2 & 1 & 1 \\ 2 & 1 & -1 \end{array}\right] \to \left[\begin{array}{c:cc} 2 & 1 & -1 \\ 2 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right] \to \left[\begin{array}{c:cc} 2 & 1 & -1 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right] \qquad \vec{r}_1 = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$$

$$\left[\begin{array}{cc:c} 1 & -1 & 2 \\ 1 & 1 & 18 \\ 1 & 1 & 18 \\ 1 & -1 & 2 \end{array}\right] \to \begin{bmatrix} 1 & -1 & 2 \\ 1 & 1 & 18 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & -1 & 2 \\ 0 & 2 & 16 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad \vec{r}_2 = \begin{bmatrix} 10 \\ 8 \end{bmatrix}$$

$$R = \begin{bmatrix} 2 & 10 \\ 0 & 8 \end{bmatrix}$$

P Find orthonormal vectors $\vec{u}_1$ and $\vec{u}_2$ in the subspace $V = \text{span}\left(\begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix}, \begin{bmatrix} 2 \\ 7 \\ -8 \end{bmatrix}\right)$.

P: Find a 2×2 matrix R such that $\begin{bmatrix} 2 & 2 \\ 1 & 7 \\ -2 & 8 \end{bmatrix} = \frac{1}{3}\begin{bmatrix} 2 & -2 \\ 1 & 2 \\ -2 & -1 \end{bmatrix} R$