Section 4.1 The Division Algorithm (Part 1)

The Division Algorithm

P 1. Find integers q and r such that $51 = 7q + r$ where $0 \leq r < 7$.

P 2. The Division Algorithm: Let $a, b \in \mathbb{Z}, b \neq 0$. Prove there exist unique integers q and r, with $0 \leq r < |b|$ such that $a = qb + r$.

P 3. Find integers q and r, with $0 \leq r < 20$ such that $-3,315 = 20q + r$.

P 4. Using the division algorithm, show that if $x \in \mathbb{Z}$ then $x = 2k$ for some integer k or $x = 2k + 1$ for some integer k.

Section 4.2 Divisibility and Euclidean Algorithm (Part 1)

Divisibility

P 5. Let a and b be integers. Prove that if $a + 5b$ is divisible by 7 then $10a + b$ is divisible be 7.

P 6. Suppose a, b, and c are integers such that $c \mid a$ and $c \mid b$. Show that $c \mid (ax + yb)$ for any integers x and y.

GCD

P 7. Prove that integers a and b have at most one greatest common divisor.

Section 4.2 Divisibility and Euclidean Algorithm (Part 2)

Euclidean Algorithm

P 8. Find integers m and n such that $-19m + 119n = 1$.

P 9. Find integers m and n such that $210m - 60n = 30$.

Section 5.1 Mathematical Induction (Part 1)

Principle of Mathematical Induction

P 10. Prove that $1 + 5 + 9 + \cdots + (4n - 3) = 2n^2 - n$ is true for all positive integers n.

P 11. Prove that $1^2 + 2^2 + 3^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}$ for $n \geq 1$.

P 12. Suppose a rectangle is subdivided into regions by means of straight lines each extending from one border of the rectangle to another. Prove that the regions of the “map” so obtained can be colored with just two colors in such a way that bordering “countries” have different colors.

Section 5.2 Recursively Defined Sequences (Part 1)

Strong Principle of Mathematical Induction

P 13. Let $a_1 = 1, a_2 = 3$ and $a_n = 2a_{n-1} - a_{n-2}$ for $n \geq 3$. Conjecture a solution for the recurrence relation and prove it using Strong Mathematical Induction.

P 14. Let $a_0 = 1$, $a_1 = 4$, and $a_n = 4a_{n-1} - 4a_{n-2}$ for $n \geq 2$. Prove that $a_n = 2^n(n + 1)$ for all $n \geq 0$.

P 15. Prove ${}^\forall n \geq 12\ {}^\exists s, t \in \mathbb{N} \cup \{0\}$ such that $n = 3s + 7t$.

Section 6.1 The Principle of Inclusion-Exclusion (Part 1)

The Cardinality of the Union of two Sets

P 16. Among the 30 students registered for a course in discrete mathmematics, 15 people know the JAVA programming language, 12 know HTML, and 5 know both of these languages.

(a) How many students know at least on of JAVA or HTML?

(b) How many students know only JAVA?

(c) How many know only HTML?

(d) How many know exactly one of the languages JAVA and HTML?

A Generalization

P 17. How many integers in [100] are not divisible by $5, 7$, or 9?

P 18. How many integers between 1 and 500 are divisible by 3 but not by 5 or 6?

P 19. How many integers between 1 and $10,000$ are divisible by 3 and 7 but not by either 5 or 11?

Section 6.2 The Addition and Multiplication Rules (Part 1)

Counting

P 20. How many ways can a password of length 5 be created using only 3 letters x, y, z?

P 21. From a group of 14 dogs, 5 cats, and 3 monkeys how many ways can a dog, a cat, and a monkey be selected?

P 22. How many three-digit numbers contain the digits 2 and 5?

P 23. Let $A = \{1, 2, \ldots, n\}$ and $B = \{1, 2\}$. Show that there are 2^n functions from A to B.

Section 6.3 The Pigeonhole Principle (Part 1)

Existence Problems

P 24. There are 15 students in a discrete math class. Show that there are at least 2 students that have birthdays in the same month. In this problem what are the Pigeons and what are the holes?

P 25. Thirty buses are to be used to transport 2000 students from South Hadley to Amherst. Each bus has 80 seats. Assume one seat per passenger.

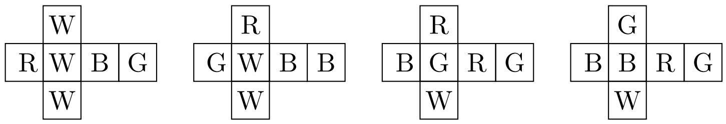
1. Prove that one of the buses will carry at least 67 passengers.

2. Prove that one of the buses will have at least 14 empty seats.

Section 9.1 A Gentle Introduction (Part 1)

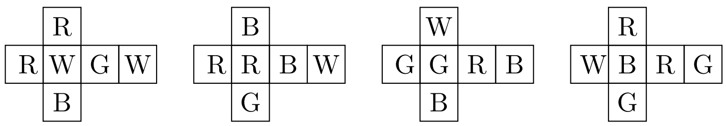
Applications

P 26. Find the Instant Insanity solution for the following cubes. (A solution exists.)



Cube 1 Cube 2 Cube 3 Cube 4

P 27. Find the Instant Insanity solution for the following cubes.



Cube 1 Cube 2 Cube 3 Cube 4