Reading Questions 10

page 149 Principle 5.1.1

page 149 Problem 3

1. The Principle of Mathematical Induction can be used to prove a statement is true for all positive integers. T
2. The Principle of Mathematical Induction could be used to prove a statement is true for all integers in the set $\{21, 22, 23, 24, \dots\}$. T
3. What is the value of $\sum_{i=1}^{3}(2i-1)$? 9

Section 5.1 Mathematical Induction (Part 1)

Principle of Mathematical Induction

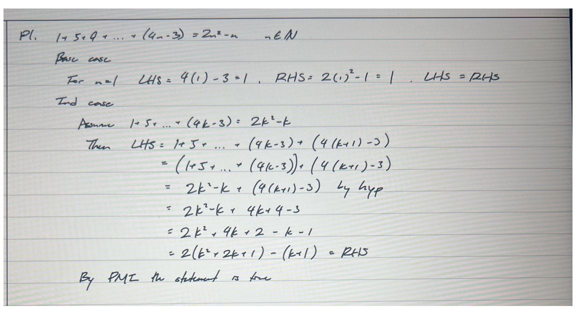
P 1. Prove that $1+5+9+\cdots+(4n-3)=2n^2-n$ is true for all positive integers n.

P 2. Prove that $1^2+2^2+3^2+\cdots+n^2=\frac{n(n+1)(2n+1)}{6}$ for $n \geq 1$.

P 3. Compute $\sum_{i=0}^{3} 2i+1$ and $\sum_{i=0}^{3} 1$.

P 4. Prove that $\sum_{i=1}^{n}(i+1)2^i = n2^{n+1}$ for $n \in \mathbb{N}$.

P 5. Suppose a rectangle is subdivided into regions by means of straignt lines each extending from one border of the rectangle to another. Prove that the regions of the "map" so obtained can be colored with just two colors in such a way that bordering "countries" have different colors.



P1. $1+5+9+\dots+(4n-3)=2n^2-n$ $\quad n\in\mathbb{N}$

Base case

For $n=1$ $\quad LHS = 4(1)-3=1$, $RHS = 2(1)^2-1=1$. $LHS=RHS$

Ind case

Assume $1+5+\dots+(4k-3)=2k^2-k$

Then $LHS = 1+5+\dots+(4k-3)+(4(k+1)-3)$

$= (1+5+\dots+(4k-3))+(4(k+1)-3)$

$= 2k^2-k+(4(k+1)-3)$ by hyp

$= 2k^2-k+4k+4-3$

$= 2k^2+4k+2-k-1$

$= 2(k^2+2k+1)-(k+1) = RHS$

By PMI the statement is true

5.1 Induction

Thm: Principle of Mathematical Induction PMI aka weak induction

Let $A = \{n_0, n_1, \dots\}$ be an infinitely countable set.

Let $P(n)$ be a statement where $n \in A$. Suppose

(1) $P(n_0)$ is true

(2) $P(n_k) \Rightarrow P(n_{k+1})$ where $k \geq 0$.

Then $P(n_i)$ is true $\forall n_i \in A$.

Ex: For all $n \in \mathbb{N}$ the statement $P(n)$ is true where

$$P(n): 1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}.$$

$A = \mathbb{N}$ $\quad n_0 = 1$

pf: (Base step $P(n_0)$ is true)

For $n = 1$, $1 = LHS$ and $RHS = \frac{1(1+1)}{2} = 1$

$LHS = RHS \Rightarrow P(1)$ is true.

(Induction Step $P(n_k) \Rightarrow P(n_{k+1})$)

Let $1 + 2 + \cdots + k = \frac{k(k+1)}{2}$ where $k \geq 1$.

Then $LHS = 1 + 2 + 3 + \cdots + k + k+1$

$= (1 + 2 + 3 + \cdots + k) + (k+1)$

$= \frac{k(k+1)}{2} + (k+1)$ by hyp.

$= \frac{k(k+1) + 2(k+1)}{2}$

$= \frac{(k+1)(k+2)}{2} = \frac{(k+1)((k+1)+1)}{2} = RHS$.

By PMI the statement is true. $\square$

<u>Ex:</u> Prove that for any integer $n \geq 1$ the sum of the odd integers from 1 to $2n-1$ is n^2.

<u>pf:</u> $A = \mathbb{N}$ $n_0 = 1$

<u>Base step</u>

For $n = 1$, $LHS = 1$ and $RHS = 1^2 = 1$. $LHS = RHS$

<u>Ind step</u>

Assume 1 to $2k-1$ $= k^2$.

Then $LHS = 1 + 3 + \cdots + 2k-1 + 2(k+1) - 1$

$= (1 + 3 + \cdots + (2k-1)) + 2(k+1) - 1$

$= k^2 + 2(k+1) - 1$ by hyp

$= k^2 + 2k + 1 = (k+1)^2 = RHS.$

By PMI the statement is true. $\square$

<u>Ex:</u> Prove $\displaystyle\sum_{t=0}^{n} 2^t = 2^{n+1} - 1$ for $n \geq 1$.

<u>pf:</u> $A = \mathbb{N}$ $n_0 = 1$

<u>Base step</u>

For $n = 1$, $LHS = 2^0 + 2^1 = 3$ and

$RHS = 2^{1+1} - 1 = 3$. $LHS = RHS$.

<u>Ind Step</u>

Assume $\displaystyle\sum_{t=0}^{k} 2^t = 2^{k+1} - 1$ where $k \geq 1$.

Then $LHS = \displaystyle\sum_{t=0}^{k+1} 2^t = \sum_{t=0}^{k} 2^t + 2^{k+1}$

$= 2^{k+1} - 1 + 2^{k+1}$ by hyp

$= 2 \cdot 2^{k+1} - 1$

$= 2^{1+k+1} - 1$

$= 2^{(k+1)+1} - 1 = RHS.$

By PMI the statement is true. $\square$

Ex: Prove that for every integer $n \geq 2$ the number of intersecting points obtained by placing n lines in the plane such that no 3 lines intersect at the same point and no 2 lines are parallel is $\frac{n(n-1)}{2}$.

Pf: $A = \mathbb{N} \setminus \{1\}$ $\quad n_0 = 2$

Base step

Let $n = 2$. Then the number of intersecting points between two lines that intersect is 1.

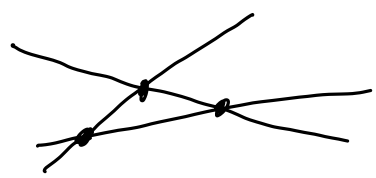
Also $\frac{2(2-1)}{2} = 1$.

Ind:

Assume The number of intersecting points between k lines is $\frac{k(k-1)}{2}$.

Put $k+1$ lines in plane. Delete a line. We have k lines in the plane. By hyp there are $\frac{k(k-1)}{2}$ intersecting points. By inserting the $k+1$ line we get

$$\frac{k(k-1)}{2} + k$$



$$= \frac{k(k-1)}{2} + \frac{2k}{2}$$

$$= \frac{k(k-1+2)}{2}$$

$$= \frac{k(k+1)}{2} = \text{RHS}.$$

By PMI the statement is true.