Section 4.2 Divisibility and Euclidean Algorithm (Part 1)

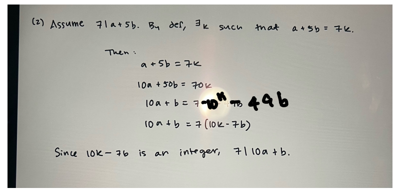
Divisibility

$369 = 3 \cdot 123$

P 1. Determine if the number 369 divides 3. Is 369 divisible by 3? yes

P 2. Let a and b be integers. Prove that if $a + 5b$ is divisible by 7 then $10a + b$ is divisible be 7.

P 3. Suppose a, b, and c are integers such that $c \mid a$ and $c \mid b$. Show that $c \mid (ax + yb)$ for any integers x and y.



(2) Assume $7 \mid a+5b$. By def, $\exists k$ such that $a + 5b = 7k$.

Then:

$a + 5b = 7k$

$10a + 50b = 70k$

$10a + b = 70k - 49b$

$10a + b = 7(10k - 7b)$

Since $10k - 7b$ is an integer, $7 \mid 10a + b$.

$70 = 2 \cdot 35 = 2 \cdot 5 \cdot 7$

$42 = 2 \cdot 21 = 2 \cdot 3 \cdot 7$

GCD

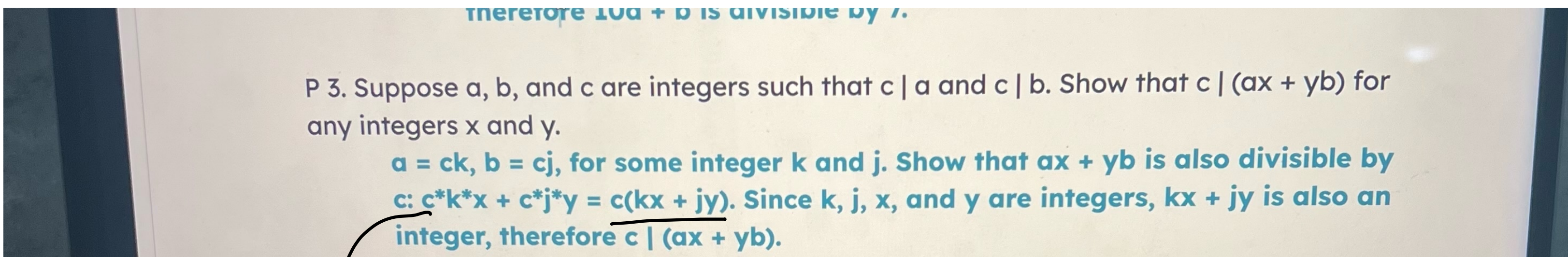
P 4. What is the greatest common divisor of 70 and 42? 14

P 5. Suppose a is a nonzero integer. What is ~~$\gcd(a, 0)$~~? DNE

P 6. Prove that integers a and b have at most one greatest common divisor.

P 7. What is the least common multiple of 7 and 13? 91

P 8. If $a \in \mathbb{Z}$ then $\gcd(a, a+2) = \begin{cases} 1 & a\text{-odd} \\ 2 & a\text{-even} \end{cases}$



therefore 10a + b is divisible by 7.

P 3. Suppose a, b, and c are integers such that c | a and c | b. Show that c | (ax + yb) for any integers x and y.

a = ck, b = cj, for some integer k and j. Show that ax + yb is also divisible by c: c*k*x + c*j*y = c(kx + jy). Since k, j, x, and y are integers, kx + jy is also an integer, therefore c | (ax + yb).

$ax + yb$

4.2 **Def:** $a, b \in \mathbb{Z}$ $b \neq 0$. Then a is divisible by b, denoted by $b \mid a$ iff $\exists k \in \mathbb{Z}$ such that $a = bk$.

Also b divides a.

Ex: 24 is divisible by 6 since $24 = 6 \cdot 4$.

Ex: Let $a \in \mathbb{Z}$ such that $a \neq 0$. Then

$a^2 \mid a^5$ since $a^5 = a^2 \cdot a^3$.

Thm: $a, b \in \mathbb{Z}$ with $b \neq 0$. If

$b \mid a$ then $b \mid (-a)$ and $-b \mid a$.

pf: WTS

If $b \mid a$ then $b \mid -a$

Assume $b \mid a$. By def, $\exists k \in \mathbb{Z}$ such that

$a = bk$. Hence $-a = b(-k)$. Since $-k \in \mathbb{Z}$

by def $b \mid -a$.

WTS

If $b \mid a$ then $-b \mid a$.

Try on your own.

Thm: For every integer n, $3 \mid n^3 - n$.

pf:

$$n^3 - n = n(n^2 - 1) = n(n-1)(n+1)$$

$3 \mid n$ or $3 \mid n-1$ or $3 \mid n+1$

$n = 3q + r$ where $0 \le r < 3 \Rightarrow r \in \{0, 1, 2\}$

$r = 0 \Rightarrow n = 3q \Rightarrow 3 \mid n$

$r = 1 \Rightarrow n - 1 = 3q \Rightarrow 3 \mid n - 1$

$r = 2 \Rightarrow n - 2 = 3q \Rightarrow n - 2 + 3 = 3q + 3$

$\Rightarrow n + 1 = 3(q + 1)$

$\Rightarrow 3 \mid n + 1$

Def: $a, b \in \mathbb{Z}$, $b \neq 0$. The greatest common divisor, $\gcd(a,b)$, is the largest common divisor of a and b.

Ex: $\gcd(4, 16) = 4$

1) $4 \mid 4$ $\quad 4 \mid 16$ $\quad$ 4 is a common divisor

2) if $\underline{c \mid 4}$ and $c \mid 16$ then $c \mid 4$. or $c \le 4$

$4 = ck \Rightarrow$

$\gcd(4, 6) = 2$

1) $2 \mid 4$ $\quad 2 \mid 6$ $\quad$ 2 is a common divisor

2) if $c \mid 4$ and $c \mid 6$ then $(c \mid 2$ or $c \leq 2)$.

Def: $a, b \in \mathbb{Z}$ $\quad b \neq 0$. The least common multiple of a and b is the smallest positive multiple of a and b. $\operatorname{lcm}(a, b)$

Ex: $\operatorname{lcm}(5, 11) = 55$

$5 \cdot 11 = 55$ and $11 \cdot 5 = 55$ $\Rightarrow$ 55 is a multiple of 5 & 11