# Reading Questions 9

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1. The binary relation $\leq$ is the relation $\{(a, b)|a, b \in \mathbb{R} \text{ and } a \leq b\}$. T
2. A relation $R$ is transitive if $(a, b), (b, c) \in R$ then $(a, c) \in R$. T
3. What is the additive inverse of 5? -5

# Section 4.1 The Division Algorithm (Part 1)

## Well-Ordering Principle

**P 1.** Which of the following sets contains a least element?

$$(a)\, \mathbb{N} \quad (b)\, \mathbb{Z} \quad (c)\, (2, 8]$$

✓ X X

## The Division Algorithm

**P 2.** Find integers $q$ and $r$ such that $51 = 7q + r$ where $0 \leq r < 7$. q = 7 r = 2

~~**P 3.**~~ **The Division Algorithm:** Let $a, b \in \mathbb{Z}, b \neq 0$. Prove there exist unique integers $q$ and $r$, with $0 \leq r < |b|$ such that $a = qb + r$.
Hint: Try using proof by cases.

- Case 1: $a = 0, b \neq 0$.
- Case 2: $a, b > 0$.
- Case 3: $a > 0, b < 0$.
- Case 4: $a < 0, b > 0$.
- Case 5: $a < 0, b < 0$.

**P 4.** Find integers $q$ and $r$, with $0 \leq r < 20$ such that $-3,315 = 20q + r$. q=-166 r=5

**P 5.** Find integers $q$ and $r$, with $0 \leq r < 20$ such that $3,315 = -20q + r$. q=-165 r=15

**P 6.** Using the division algorithm, show that if $x \in \mathbb{Z}$ then $x = 2k$ for some integer $k$ or $x = 2k + 1$ for some integer $k$.

4.1

**Def:** Let $\emptyset \neq A \subseteq \mathbb{R}$ and $x \in A$. Then $x$ is the least element of $A$ if $x \leq b$ for all $b \in A$.

**Ex:** $\{x \in \mathbb{Z} \mid (x-2)(x-0.5)(x+3)=0\}$ has a least element which is $-3$. $\mathbb{R}$ does not have a least.

Assume $a$ is the least element.

$a-1 \in \mathbb{R}$ and $a-1 < a \Rightarrow \Leftarrow$.

**Def:** Let $S \subseteq A$ where $S \neq \emptyset$. Then $A$ is well-ordered if $S$ has a least element.

**Ex:** $\underset{\{x \in \mathbb{R} : 1 \leq x < 3\}}{[1,3)}$ half-open has the least element 1 but is not well-ordered since $(1,3) \subseteq [1,3)$ and $(1,3)$ does not have a least element.

open $\{x \in \mathbb{R} : 1 < x < 3\}$

**Ex:** $\mathbb{N}$ is well-ordered.

**Thm:** There is no positive integer $x$ such that $0 < x < 1$.

**pf:** $\text{\textsection}$ $\exists a \in (0,1)$ s.t $a \in \mathbb{N}$.

Let $A$ be the set of all such $a$. $A \subseteq \mathbb{N}$. $\mathbb{N}$ is well-ordered $\Rightarrow$ $A$ has a least element $y$. Then $0 < y < 1$

$\Rightarrow 0 < y^2 < 1$. ?

Hence $y^2 \in A$. $\Rightarrow$

$$\frac{0 < y \le y^2 < 1}{0 < 1 \le y < \frac{1}{y}} \Rightarrow \Leftarrow \quad 0 < y < 1$$

(Well-Ordering Principle)

The set $\mathbb{N}$ is well-ordered.

**Lem:** Let $a, b \in \mathbb{N}$. Then there exists unique nonnegative integer $q$ and $r$ with $0 \le r < b$ such that

$$a = qb + r.$$

$q$ - quotient

$r$ - remainder

$12 = q\,14 + r$

$0 \le r < 14$

$q = 0 \quad r = 12$

$12 = 0 \cdot 14 + 12$

$\frac{12}{14}$ or $\frac{14}{12}$

**Ex:**

Find integers $q$ and $r$ such that

$-235 = -20q + r$ and $0 \le r < 20$.

$235 = 20q + (-r)$

$20q = 220 \Rightarrow 11 = q$

$235 = 220 + (-r) \Rightarrow -r = 15$

$\Rightarrow r = -15 < 0$

$20q = 240 \Rightarrow q = 12$

$235 = 240 + (-r) \Rightarrow -r = -5 \Rightarrow r = 5$