Reading Questions 7

reading: Section 9.5 : Example 3

1. The ratio test can be used to determine where a power series converges. T
2. The series $1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!} + \cdots$ converges to e^x. T
3. What does the variable R represent in the example? radius of convergence

Section 9.5 Power Series and Interval of Convergence (Part 1)

Power Series and Interval of Convergence

Definition: power series

A power series about the real number a is of the form

$$\sum_{n=0}^{\infty} C_n (x-a)^n = C_0 + C_1(x-a) + C_2(x-a)^2 + \cdots .$$

P1. $\sum_{n=1}^{\infty} \frac{1}{2^n \cdot n!}(x-0)^n$ $C_n = 2^n \cdot n!$ $a = 0$

P 1. Use $\sum$ notation to write the series

$$\frac{1}{2}x + \frac{1}{2^2 \cdot 2!}x^2 + \frac{1}{2^3 \cdot 3!}x^3 + \cdots .$$

State C_n and a.

P 2. Find the interval of convergence for the power series $\sum_{n=0}^{\infty} (\frac{x+1}{2})^n$.

P.2 Find interval of convergence for $\sum_{n=0}^{\infty} \left(\frac{x+1}{2}\right)^n$

$\sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n (x-(-1))^n$ $a=-1$

converges when $\left|\frac{x+1}{2}\right| < 1 \Rightarrow |x| < 1$?

so interval of convergence $\boxed{-1 < x < 1}$

$|x+1| < 2$

$-2 < x+1 < 2$

$-3 < x < 1$

$(-3, 1)$

Radius of Convergence

Theorem: radius of convergence

Consider the series $\sum_{n=0}^{\infty} C_n (x-a)^n$ where R is the radius of convergence and $a_n = C_n(x-a)^n$. Then

1. If $\lim_{n\to\infty} |\frac{a_{n+1}}{a_n}|$ is infinite then $R = 0$.
2. If $\lim_{n\to\infty} |\frac{a_{n+1}}{a_n}| = 0$ then $R = \infty$.
3. If $\lim_{n\to\infty} |\frac{a_{n+1}}{a_n}| = K|x-a|$ where K is a positive number then $R = \frac{1}{K}$.

P 3. Determine the radius of convergence of the series

$$\frac{(x-1)}{2} - \frac{(x-1)^2}{2 \cdot 2^2} + \frac{(x-1)^3}{3 \cdot 2^3} - \frac{(x-1)^4}{4 \cdot 2^4} + \cdots + (-1)^n \frac{(x-1)^n}{n \cdot 2^n} + \cdots .$$

State a and C_n and a_n.

P 4. When the series is not geometric, to determine the interval of convergence you must determine if the series converges at the endpoints. Determine the radius and interval of convergence of the series $\sum_{n=0}^{\infty} n^3 x^n$.

P 5. Determine the interval of convergence of the series $\sum_{n=0}^{\infty} \frac{x^{2n+1}}{n!}$. *

9.5

Def: A power series about the real number a is of the form

$$\sum_{n=0}^{\infty} c_n (x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + \dots$$

Ex: Use Σ to represent

$$1 + \frac{(x-2)^1}{4} - \frac{(x-2)^2}{9} + \frac{(x-2)^3}{16} - \frac{(x-2)^4}{25} + \dots$$

$a = 2$ $\quad (x-a) = (x-2) \quad c_n = \frac{1}{(n+1)^2}$

$$\sum_{n=0}^{\infty} \frac{1}{(n+1)^2}(x-2)^n$$

Def: The interval of convergence is the interval in which the power series converges.

Ex:

$$\sum_{n=0}^{\infty} \left(\frac{x}{2}\right)^n = \sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n (x-0)^n$$

The series is geometric so it converges when $\left|\frac{x}{2}\right| < 1$

$\Rightarrow \quad |x| < 2 \quad \Rightarrow \quad \underline{-2 < x < 2} \quad \Rightarrow \quad \underline{(-2, 2)} \leftarrow$ interval of convergence (IOC)

Def: Let R be the radius of convergence. Then $R=0$ if the series only converges at a, $R=\infty$ if the series converges everywhere, R is a positive number if the series converges on $(a-R, a+R)$.

Thm: Consider $\sum_{n=0}^{\infty} c_n(x-a)^n$, $a_n = c_n(x-a)^n$. Then

1. If $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|$ is infinite then $R=0$

2. If $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = 0$ then $R=\infty$

3. If $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = K|x-a|$ where K is a real #

   then $R = \frac{1}{K}$.

Ex: Determine the radius of convergence for the series

$$\frac{(x-1)}{3} - \frac{(x-1)^2}{2\cdot 3^2} + \frac{(x-1)^3}{3\cdot 3^3} - \frac{(x-1)^4}{4\cdot 3^4} + \cdots + (-1)^n\frac{(x-1)^n}{n\cdot 3^n} + \cdots$$

$$a_n = (-1)^n \frac{(x-1)^n}{n\cdot 3^n} \qquad a_{n+1} = (-1)^{n+1}\frac{(x-1)^{n+1}}{(n+1)\,3^{(n+1)}}$$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{\dfrac{(-1)^n(-1)\,(x-1)^n(x-1)}{(n+1)\,3^n\cdot 3}}{\dfrac{(-1)^n\,(x-1)^n}{n\cdot 3^n}}\right|$$

$$= \lim_{n \to \infty} \left| \frac{\cancel{(-1)^n}(-1)\cancel{(x-1)^n}(x-1)}{(n+1)\,\cancel{3^n}\cdot 3} \cdot \frac{n \cdot \cancel{3^n}}{\cancel{(-1)^n}\,\cancel{(x-1)^n}} \right|$$

$$= \lim_{n \to \infty} \left| \frac{-(x-1)\cdot n}{(n+1)\cdot 3} \right| = \lim_{n \to \infty} \left| \frac{-(x-1)}{3} \right| \cdot \left| \frac{n}{n+1} \right|$$

$$= \frac{|x-1|}{3} \lim_{n \to \infty} \frac{n}{n+1}$$

$$= |x-1| \cdot \frac{1}{3}$$

$K = \frac{1}{3}$ $\quad a = 1 \quad$ $R = \frac{1}{K} = 3$

IOC $(-2, 4)$ or $[-2, 4)$ or $(-2, 4]$ or $[-2, 4]$

<u>Ex:</u> Determine IOC for $\sum_{n=0}^{\infty} (-1)^n \frac{(x-1)^n}{n \cdot 3^n}$.

for $x = -2$ $\quad \sum_{n=0}^{\infty} (-1)^n \frac{(-2-1)^n}{n \cdot 3^n} = \sum_{n=0}^{\infty} (-1)^n \frac{(-3)^n}{n \cdot 3^n}$

$= \sum_{n=0}^{\infty} \frac{1}{n}$ diverges by p-series test

for $x = 4$ $\quad \sum_{n=0}^{\infty} (-1)^n \frac{(4-1)^n}{n \cdot 3^n} = \sum_{n=0}^{\infty} (-1)^n \frac{3^n}{n \cdot 3^n}$

$= \sum_{n=0}^{\infty} (-1)^n \frac{1}{n}$ converges by the alternating series test.

$\therefore$ IOC $= (-2, 4]$