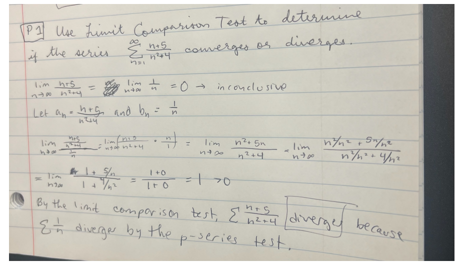
Section 9.4 Tests for Convergence (Part 2)

Limit Comparison



Theorem: Limit Comparison Test

Suppose $a_n > 0$ and $b_n > 0$ for all n. If

$$\lim_{n\to\infty} \frac{a_n}{b_n} = c$$

where $c > 0$ then the two series $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ both converge or both diverge.

P 1. Use the Limit Comparison Test to determine if the series $\sum_{n=1}^{\infty} \frac{n+5}{n^2+4}$ converges or diverges. In general, you may use any test to determine if a series converges or diverges.

P 2. Use the Limit Comparison Test to determine if the series $\sum_{n=1}^{\infty} \frac{n+2}{n^4+n+1}$ converges or diverges.

Absolute Value Test

Theorem: Absolute Value Test

If $\sum_{n=1}^{\infty} |a_n|$ converges, then so does $\sum_{n=1}^{\infty} a_n$.

P 3. Use the absolute value test to determine if the series $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^2+1}$ converges or diverges.

P 4. Determine if the series $1 - \frac{1}{4} + \frac{1}{8} - \frac{1}{16} + \frac{1}{32} - \cdots$ converges or diverges.

P 5. Is the statement "If $\sum_{n=1}^{\infty} a_n$ converges then $\sum_{n=1}^{\infty} |a_n|$ converges" always true. Explain your answer.

P 6. Suppose $|a_n| < ax^n$ where $|a_i| < a$ and $0 < x < 1$. Does the series $\sum_{n=1}^{\infty} a_n$ converge or diverge?

9.4 part 2

Thm: Limit Comparison Test

$a_n > 0$ and $b_n > 0$ for all n. If

$$\lim_{n\to\infty} \frac{a_n}{b_n} = c \quad \text{where } c \text{ is a real number and } c > 0$$

then $\sum a_n$ and $\sum b_n$ both converges or they both diverge.

Ex: Determine if $\sum_{n=1}^{\infty} \frac{n^2-5}{n^3+n+2}$ diverges or converges.

$$\lim_{n\to\infty} \frac{n^2-5}{n^3+n+2} = 0 \Rightarrow \text{we can't say the series diverges, yet!}$$

consider $a_n = \frac{n^2-5}{n^3+n+2}$ and $b_n = \frac{1}{n}$ as a_n and b_n behave in the same way. So

$$\lim_{n\to\infty} \frac{\frac{n^2-5}{n^3+n+2}}{\frac{1}{n}} = \lim_{n\to\infty} \frac{n^3-5n}{n^3+n+2}$$

$$= \lim_{n\to\infty} \frac{n^3/n^3 - 5n/n^3}{n^3/n^3 + n/n^3 + 2/n^3}$$

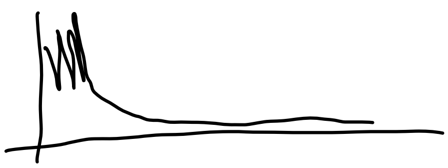
$$= \lim_{n\to\infty} \frac{1 - 5/n^2}{1 + 1/n^2 + 2/n^3}$$

$$= \frac{1-0}{1+0+0} = 1 > 0$$

By the limit comparison test $\sum a_n$ diverges since $\sum \frac{1}{n}$ diverges (by the p-series test)

Ex: Determine if $\sum \sin \frac{1}{n}$ converges or diverges.

$$\lim_{n \to \infty} \sin \frac{1}{n} = \sin(0) = 0 \Rightarrow \text{"inconclusive"}$$



$$\cancel{\sin \frac{1}{n} \overset{?}{<} \frac{1}{n^2}}$$

$$\cancel{n^2 \sin \frac{1}{n} \overset{?}{<} 1}$$

?

let $a_n = \sin \frac{1}{n}$ and $b_n = \frac{1}{n}$.

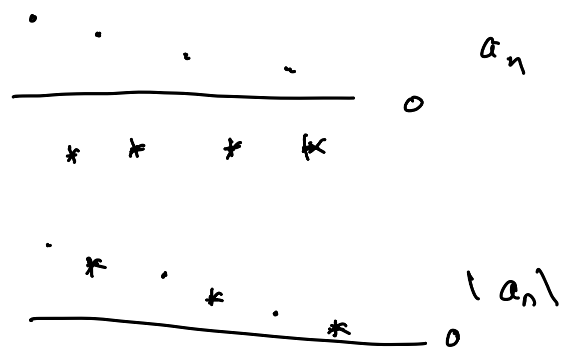
Then
$$\lim_{n \to \infty} \frac{\sin \frac{1}{n}}{\frac{1}{n}} \overset{LH}{=} \lim_{n \to \infty} \frac{\frac{-\cos \frac{1}{n}}{n^2}}{\frac{-1}{n^2}} = \lim_{n \to \infty} \cos \frac{1}{n}$$

$$= \cos 0 = 1 > 0.$$

By the limit comparison test $\sum \sin \frac{1}{n}$ diverges since $\sum \frac{1}{n}$ diverges.

9.4 absolute value test

Thm: If $\sum |a_n|$ converges then $\sum a_n$ converges.



Ex: Determine if $\sum_{n=1}^{\infty} \underbrace{3\left(-\frac{1}{2}\right)^n}_{a_n}$ converges or diverges.

$\lim a_n = 0$

We know $\sum a_n$ converges to $\dfrac{3\left(-\frac{1}{2}\right)}{1-\left(-\frac{1}{2}\right)}$

since it is geometric.

By the absolute value test

$$\sum \left|3\left(-\tfrac{1}{2}\right)^n\right| = \sum 3\left|(-1)^n\right|\left(\tfrac{1}{2}\right)^n$$

$$= \sum 3\left(\tfrac{1}{2}\right)^n.$$

Since the series is geometric and $\left|\frac{1}{2}\right| < 1$ it converges.

By absolute value test $\sum \left|3\left(-\frac{1}{2}\right)^n\right|$ converges so

$\sum 3(-\frac{1}{2})^n$ converges.

<u>Ex:</u> let $a_n = \dfrac{(-1)^n}{n^2}$.

$|\underbrace{(-1)^n}| = 1$

consider $\sum \left|\dfrac{(-1)^n}{n^2}\right| = \sum \dfrac{1}{n^2}$

$\sum \frac{1}{n^2}$ converges by p-series

$\Rightarrow$ By absolute value test $\sum \frac{(-1)^n}{n^2}$ converges.