Reading Questions 12

Section 7.6 : Example 1

1. The improper integral $\int_1^\infty \frac{1}{\sqrt{x}}\,dx$ is the limit of some sequence.
2. The sequence $\int_1^b \frac{1}{\sqrt{x}}\,dx$, grows without bound as $b \to \infty$.
3. Does $\int_1^\infty \frac{1}{\sqrt{x}}\,dx$ converge or diverge?

Section 7.6 Improper Integrals (Part 1)

Type I

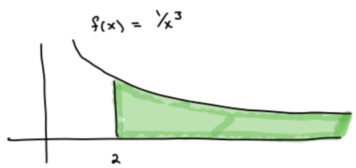
Theorem: Improper Integral I

Suppose $f(x)$ is positive for $x \geq a$. If $\lim\limits_{b\to\infty} \int_a^b f(x)\,dx$ converges then

$$\int_a^\infty f(x)\,dx = \lim_{b\to\infty} \int_a^b f(x)\,dx$$

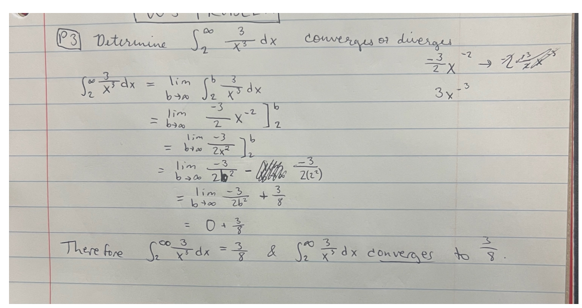
and the integral converges. Otherwise $\int_a^\infty f(x)\,dx$ diverges.

P 1. Write down an integral that represents the area of the shaded region in the figure below.



~~**P 2.** Determine if $\int_0^\infty \frac{e^x}{1+e^x}\,dx$ converges or diverges.~~

P 3. Determine if $\int_2^\infty \frac{3}{x^3}\,dx$ converges or diverges.



Type II

P 4. Determine if $\int_{-\infty}^\infty \frac{1}{x^2+25}\,dx$ converges or diverges.

Theorem: Improper Integral II

Suppose $f(x) > 0$ and continuous on $a \leq x < b$ and tends to infinity as $x \to b$. Then

$$\int_a^b f(x)\,dx = \lim_{c\to b^-} \int_a^c f(x)\,dx$$

and the integral converges. Otherwise the integral diverges.

P 5. Determine if $\int\limits_3^6 \frac{1}{(4-x)^2}\,dx$ converges or diverges.

From Mon 11-11-24

Trigonometric Substitutions

P 3. Use trig substitution to find $\int \frac{1}{\sqrt{9-x^2}}\,dx$.

perfect square

P 4. Find $\int \sqrt{16-x^2}\,dx$.

P 5. Find $\int \frac{1}{x^2+1}\,dx$ by using trig substitution.

$x = 3\sin y \qquad dx = 3\cos y\,dy$

$x^2 = (3\sin y)^2$

$1 - \sin^2 y = \cos^2 y$

$9 - 9\sin^2 y = 9\cos^2 y = (3\cos y)^2$

$$\int \frac{1}{\sqrt{9-x^2}}\,dx = \int \frac{1}{\sqrt{9-9\sin^2 y}} \cdot 3\cos y\,dy$$

$$= \int \frac{1}{\sqrt{(3\cos y)^2}} \cdot 3\cos y\,dy$$

$$= \int \frac{1}{3\cos y} \cdot 3\cos y\,dy = \int dy$$

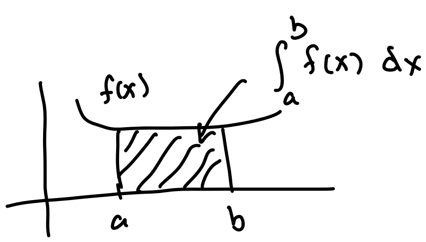
$$= \underline{y} + c_1$$

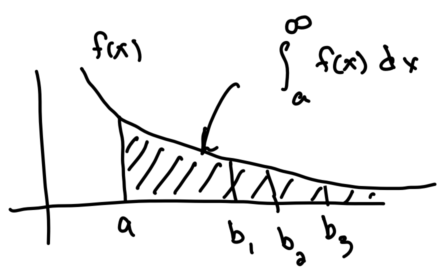
$x = 3\sin y \Rightarrow \frac{x}{3} = \sin y$

$\Rightarrow \sin^{-1}\frac{x}{3} = y$

$$= \sin^{-1}\frac{x}{3} + c_2$$

7.6 Improper Integrals





$\int_a^{b_1} f(x)\,dx, \quad \int_a^{b_2} f(x)\,dx, \quad \int_a^{b_3} f(x)\,dx, \ldots$

Def:

$$\int_a^\infty f(x)\,dx = \lim_{b\to\infty} \int_a^b f(x)\,dx$$

where $f(x) > 0$ for all x

Ex: Determine if $\int_2^\infty \frac{1}{x^2}\,dx$ converges or diverges.

If $\int_a^\infty f(x)\,dx$ exists then $\int_a^\infty f(x)\,dx$ converges to $\lim_{b\to\infty}\int_a^b f(x)\,dx$. Otherwise $\int_a^\infty f(x)\,dx$ diverges.

$$\lim_{b\to\infty}\int_2^b \frac{1}{x^2}\,dx = \lim_{b\to\infty}\left(-\frac{1}{x}\Big]_2^b\right)$$

$$= \lim_{b\to\infty} -\frac{1}{b}+\frac{1}{2}$$

$$= \lim_{b\to\infty} -\frac{1}{b} + \lim_{b\to\infty}\frac{1}{2}$$

$$= 0 + \frac{1}{2} = \frac{1}{2}$$

Therefore $\int_2^\infty \frac{1}{x^2}\,dx = \frac{1}{2}$ and $\int_2^\infty \frac{1}{x^2}\,dx$ converges to $\frac{1}{2}$.

Ex: Determine if $\int_{-\infty}^\infty \frac{e^x}{1+e^x}\,dx$ converges or diverges

$$\int_{-\infty}^\infty \frac{e^x}{1+e^x}\,dx = \int_{-\infty}^0 \frac{e^x}{1+e^x}\,dx + \int_0^\infty \frac{e^x}{1+e^x}\,dx$$

$$\stackrel{?}{=} \lim_{c\to-\infty}\int_c^0 \frac{e^x}{1+e^x}\,dx + \lim_{b\to\infty}\int_0^b \frac{e^x}{1+e^x}\,dx$$

$u = 1+e^x$

$du = e^x du$

$$= \lim_{c\to-\infty}\int_c^0 \frac{du}{u} + \lim_{b\to\infty}\int_0^b \frac{du}{u}$$

$$= \lim_{c\to-\infty} \ln|1+e^x|\Big]_c^0 + \lim_{b\to\infty} \ln|1+e^x|\Big]_0^b$$

$$= \lim_{c \to -\infty} \ln|1+e^0| - \ln|1+e^c|$$

$$+ \lim_{b \to \infty} \ln|1+e^b| - \ln|1+e^0|$$

$$= \underbrace{\lim_{c \to -\infty} \ln 2 - \ln|1+e^c|}_{} + \underbrace{\lim_{b \to \infty} \ln|1+e^b| - \ln 2}_{\text{DNE}}$$

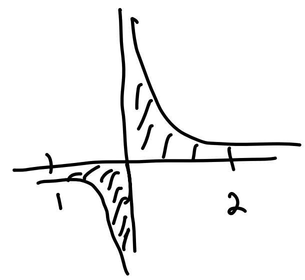
$$= \ln 2 - \ln 1 + \dots$$

$\Rightarrow$ Since $\lim_{b \to \infty} \ln|1+e^b|$ DNE, $\int_{-\infty}^{\infty} \frac{e^x}{1+e^x}\,dx$ diverges.

<u>Ex:</u> Determine if $\int_{-1}^{2} \frac{1}{x^4}\,dx$ converges or diverges.

improper as $\frac{1}{x^4}$ DNE always at $[-1, 2]$

$$\int_{-1}^{2} \frac{1}{x^4}\,dx = \int_{-1}^{0} \frac{1}{x^4}\,dx + \int_{0}^{2} \frac{1}{x^4}\,dx$$



$$= \lim_{c \to 0^-} \int_{-1}^{c} \frac{1}{x^4}\,dx + \lim_{c \to 0^+} \int_{c}^{2} \frac{1}{x^4}\,dx$$

$\int x^{-4}\,dx = \frac{x^{-4+1}}{-4+1}$

$= \frac{1}{3x^3}$

$$= \lim_{c \to 0^-} \left. \frac{-3^{-1}}{x^3} \right]_{-1}^{c} + \lim_{c \to 0^+} \left. \frac{-3^{-1}}{x^3} \right]_{c}^{2}$$

$$= \lim_{c \to 0^-} \frac{-3^{-1}}{c^3} - \underline{\frac{-3^{-1}}{(-1)^3}} + \lim_{c \to 0^+} \underline{\frac{-3^{-1}}{2^3}} - \frac{-3^{-1}}{c^3}$$

$$\lim_{c \to 0^-} \frac{-3^{-1}}{c^3} \to \infty$$

$$\lim_{c \to 0^+} \frac{3^{-1}}{c^3} \to \infty$$

This implies $\int_{-1}^{2} \frac{1}{x^4}\,dx$ diverges.