# Section 5.4 Theorems about Definite Integrals (Part 1)

## Properties of the Definite Integral

**P 1.** Compute $\int_{\pi}^{-\pi} \cos(x)\,dx$.

**P 2.** If $\int_{10}^{20} f(x)\,dx = 9$ and $\int_{20}^{13} f(x)\,dx = 4$ what is $\int_{10}^{13} f(x)\,dx$?

$\Rightarrow \int_{13}^{20} = -4$

$\int_{10}^{20} = \int_{10}^{13} \int_{13}^{20} \Rightarrow \int_{10}^{13} = \int_{10}^{20} - \int_{13}^{20}$

$9 - (-4)$

$= 13$

**P 3.** Compute $\int_{2}^{1} f(x)\,dx$ and $\int_{2}^{1} 5g(x) + 3f(x)\,dx$ given that $\int_{1}^{2} f(x)\,dx = 10$ and $\int_{1}^{2} g(x)\,dx = 5$.

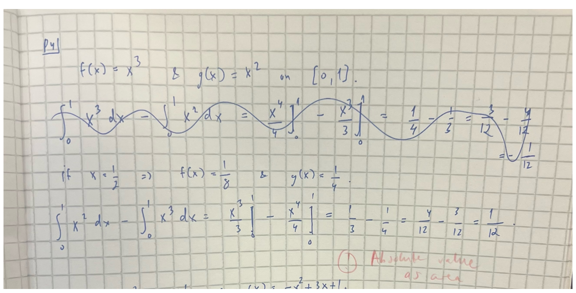
$-10$

**P 4.** Find the area of the region bounded by $f(x) = x^3$ and $g(x) = x^2$ on $[0, 1]$.

**P 5.** Find the area of the region bounded between $f(x) = x^2 + x + 1$ and $g(x) = -x^2 + 3x + 1$.

**P 6.** What is the average value of the function $f(x) = 1 + x$ on the interval $[0, 2]$?

**P 7.** Let $\int_a^b f(x)\,dx = 8$. Find $\int_{a+5}^{b+5} f(x-5)\,dx$.



P4

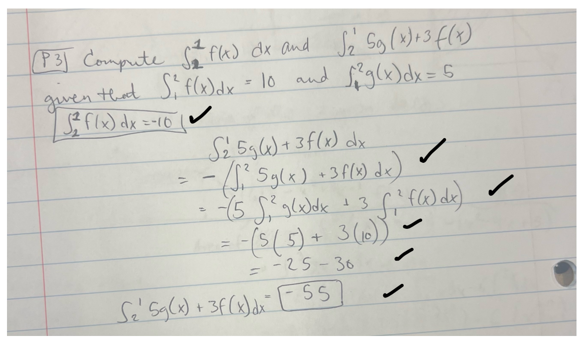
$f(x) = x^3$ & $g(x) = x^2$ on $[0, 1]$.

$\int_0^1 x^3\,dx - \int_0^1 x^2\,dx = \frac{x^4}{4}\Big]_0^1 - \frac{x^3}{3}\Big]_0^1 = \frac{1}{4} - \frac{1}{3} = \frac{3}{12} - \frac{4}{12} = -\frac{1}{12}$

if $x = \frac{1}{2} \Rightarrow f(x) = \frac{1}{8}$ & $g(x) = \frac{1}{4}$.

$\int_0^1 x^2\,dx - \int_0^1 x^3\,dx = \frac{x^3}{3}\Big]_0^1 - \frac{x^4}{4}\Big]_0^1 = \frac{1}{3} - \frac{1}{4} = \frac{4}{12} - \frac{3}{12} = \frac{1}{12}$.

(!) Absolute value as area



P3 Compute $\int_2^1 f(x)\,dx$ and $\int_2^1 5g(x)+3f(x)$ given that $\int_1^2 f(x)dx = 10$ and $\int_1^2 g(x)dx = 5$

$\boxed{\int_2^1 f(x)\,dx = -10}$

$\int_2^1 5g(x) + 3f(x)\,dx$

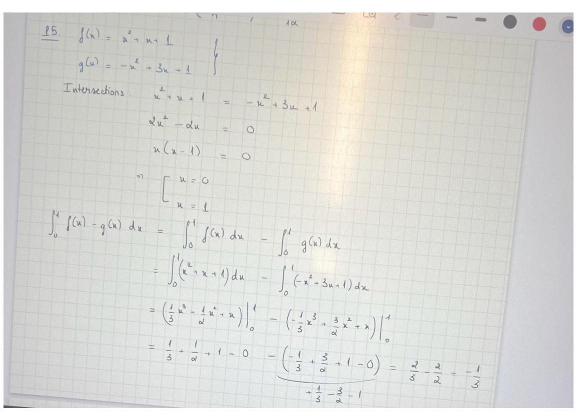
$= -\left(\int_1^2 5g(x) + 3f(x)\,dx\right)$

$= -\left(5\int_1^2 g(x)dx + 3\int_1^2 f(x)\,dx\right)$

$= -\left(5(5) + 3(10)\right)$

$= -25 - 30$

$\int_2^1 5g(x) + 3f(x)dx = \boxed{-55}$



P5. $f(x) = x^2 + x + 1$

$g(x) = -x^2 + 3x + 1$

Intersections: $x^2 + x + 1 = -x^2 + 3x + 1$

$2x^2 - 2x = 0$

$x(x-1) = 0$

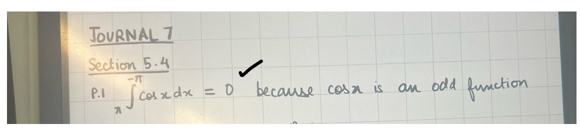
$\Rightarrow x = 0$, $x = 1$

$\int_0^1 f(x) - g(x)\,dx = \int_0^1 f(x)\,dx - \int_0^1 g(x)\,dx$

$= \int_0^1 (x^2 + x + 1)\,dx - \int_0^1 (-x^2 + 3x + 1)\,dx$

$= \left(\frac{1}{3}x^3 + \frac{1}{2}x^2 + x\right)\Big|_0^1 - \left(-\frac{1}{3}x^3 + \frac{3}{2}x^2 + x\right)\Big|_0^1$

$= \frac{1}{3} + \frac{1}{2} + 1 - 0 - \underbrace{\left(-\frac{1}{3} + \frac{3}{2} + 1 - 0\right)}_{+\frac{1}{3} - \frac{3}{2} - 1} = \frac{2}{3} - \frac{2}{2} = -\frac{1}{3}$



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Section 5.4

P.1 $\int_{\pi}^{-\pi} \cos x\,dx = 0$ because $\cos x$ is an odd function

5.4

$a, b, c$ are real numbers $f, g, h$ are continuous

**Thm:**

$$\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$$

**Ex:** Let $\int_0^{1.25} \cos(x^2)\,dx = 0.98$. Then

$$\int_{1.25}^{0} \cos(x^2)\,dx = -\int_0^{1.25} \cos(x^2)\,dx = -0.98.$$

**Thm:**

$a \le b \le c$

$$\int_a^c f(x)\,dx = \int_a^b f(x)\,dx + \int_b^c f(x)\,dx$$

**Ex:** Let $\int_0^{1.25} \cos(x^2)\,dx = 0.98$ and $\int_0^1 \cos x^2\,dx = 0.90$.

$$\int_1^{1.25} \cos(x^2)\,dx = \int_0^{1.25} \cos(x^2)\,dx - \int_0^1 \cos(x^2)\,dx$$

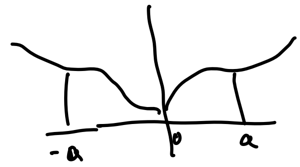
$$= 0.98 - 0.90$$

and

$$\int_{-1}^{1} \cos(x^2)\,dx = \int_{-1}^{0} \cos x^2\,dx + \int_0^1 \cos x^2\,dx$$

Note $f(x)$ is even if $f(-x) = f(x)$

$f(x)$ is odd if $f(-x) = -f(x)$




$$= \int_0^1 \cos x^2 \, dx + \int_0^1 \cos x^2 \, dx$$

$$= (0.90)\,2$$

Thm

$$\int_a^b c f(x) \pm g(x) \, dx = c\int_a^b f(x)\, dx \pm \int_a^b g(x)\, dx$$

Ex: Find $\int_0^3 (1 + 4x)\, dx$.

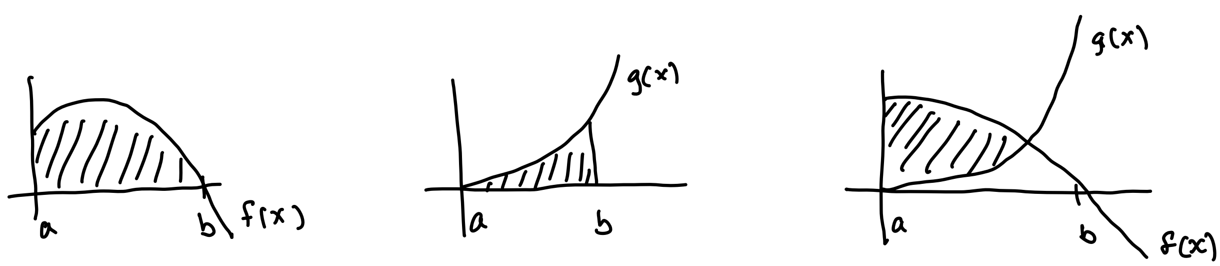
$$\int_0^3 1 + 4x \, dx = \int_0^3 1 \, dx + 4\int_0^3 x \, dx$$

$$= x\Big]_0^3 + 4\left( \frac{x^2}{2}\Big]_0^3 \right)$$

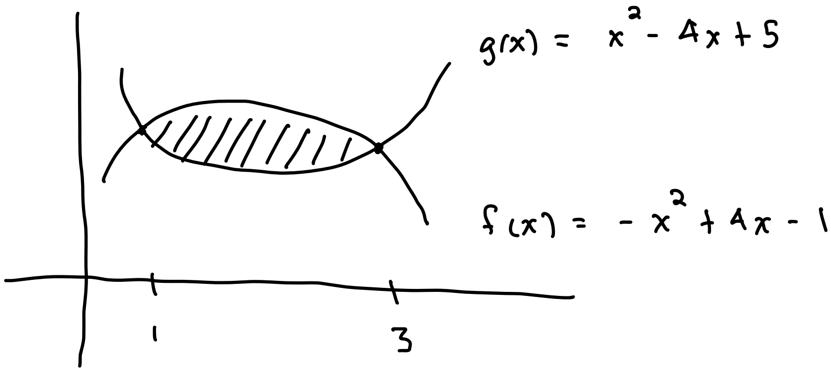
$$= 3 + 4\left(\frac{3^2}{2}\right) = 3 + 18 = 21$$

Thm:

Area between $f$ and $g$ for $a \le x \le b$ where $g(x) \le f(x)$ $= \int_a^b f(x) - g(x) \, dx$



Ex: Find the area of the region below.




$g(x) = f(x)$ $\qquad$ $x^2 - 4x + 5 = -x^2 + 4x - 1$

$$2x^2 - 8x + 6 = 0$$

$$x^2 - 4x + 3 = 0$$

$$(x-3)(x-1) = 0$$

$$x = 1, 3$$

area of //// $= \displaystyle\int_1^3 \overbrace{(-x^2 + 4x - 1)}^{f(x)} - \overbrace{(x^2 - 4x + 5)}^{g(x)} \, dx$

$$= \int_1^3 -2x^2 + 8x - 6 \, dx$$

$$= -2\int_1^3 x^2 \, dx + 8\int_1^3 x \, dx - 6\int_1^3 1 \, dx$$

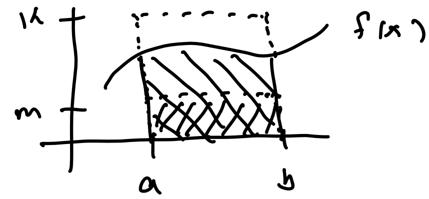
$$= -2\left( \frac{x^3}{3} \Big|_1^3 \right) + 8\left( \frac{x^2}{2} \Big]_1^3 \right) - 6\left( x \Big]_1^3 \right)$$

$$= -2\left( \frac{3^3}{3} - \frac{1^3}{3} \right) + 8\left( \frac{3^2}{2} - \frac{1^2}{2} \right) - 6(3-1)$$

Thm:

$$0 \le m \le f(x) \le k \quad \Rightarrow \quad m(b-a) \le \int_a^b f(x)\,dx \le k(b-a)$$

$$0 \le a \le x \le b$$




$$\begin{gathered} 0 \le f(x) \le g(x) \\ a \le x \le b \end{gathered} \quad \Rightarrow \quad \int_a^b f(x)\,dx \le \int_a^b g(x)\,dx$$

Thm:

Average value of f from a to b $= \dfrac{1}{b-a} \displaystyle\int_a^b f(x)\,dx$